

Chip-Firing and Riemann-Roch Theory for Directed Graphs

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Abstract

We investigate Riemann-Roch theory for directed graphs. The Riemann-Roch criteria of Amini and Manjunath is generalized to all integer lattices orthogonal to some positive vector. Using generalized notions of a v_0 -reduced divisor and Dhar's algorithm we investigate two chip-firing games coming from the rows and columns of the Laplacian of a strongly connected directed graph. We discuss how the “column” chip-firing game is related to directed \tilde{G} -parking functions and the “row” chip-firing game is related to the sandpile model. We conclude with a discussion of arithmetical graphs, which after a simple transformation may be viewed as a special class of directed graphs which will always have the Riemann-Roch property for the column chip-firing game. Examples of arithmetical graphs are provided which demonstrate that either, both, or neither of the two Riemann-Roch conditions may be satisfied for the row chip-firing game.

Contents

1	Introduction	2
1.1	Basic Notations and Definitions	4
2	Riemann-Roch Theory for Sub-lattices of Λ_R	5
2.1	Preliminaries	5
2.2	Extreme Points of $\Sigma(\Lambda)$ and $\bar{\Sigma}_{\mathbb{R}}(\Lambda)$	6
2.3	Riemann-Roch Theorem for Uniform and Reflection Invariant Sub-lattices of Λ_R	9
2.4	Riemann-Roch Theorem for sub-lattice of Λ_R and $\Lambda_{\bar{I}}$	11
3	Chip-Firing Game on Directed Graphs	12
3.1	Row Chip-Firing Game, The Sandpile Model and Riemann-Roch Theory	12
3.1.1	Reduced Divisors	13
3.1.2	Dhar's Algorithm	15
3.1.3	The Sandpile Model	15
3.2	Column Chip-Firing Game, \tilde{G} -Parking Functions, and Riemann-Roch Theory	17

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4	Arithmetical Graphs	18
4.1	Arithmetical Graphs with the Riemann-Roch Property	23
4.2	Arithmetical Graphs without the Riemann-Roch Property	26
	Acknowledgments	28
	References	28

1 Introduction

We use standard graph theoretic and discrete geometric notation and terminology, which may be found in [6, 11] and [20, 26] respectively. All graphs in this paper are finite and have no loops although they may be directed or have multiple edges (multi-graphs). We refer the reader to Section 1.1 for some basic notation and definitions.

Let R be a positive $n + 1$ dimensional vector and $\Lambda_R = \{D \in \mathbb{Z}^{n+1} : D \cdot R = 0\}$. Fix Λ , a full-dimensional sub-lattice of Λ_R . As noted in 1.1, we refer to an element $D \in \mathbb{Z}^{n+1}$ as a divisor. We say divisors $D, D' \in \mathbb{Z}^{n+1}$ are *equivalent*, denoted by $D \sim D'$, if and only if $D - D' \in \Lambda$. We say a divisor $E \in \mathbb{Z}^{n+1}$ is *effective* if $E \geq \vec{0}$. For any divisor $D \in \mathbb{Z}^{n+1}$, the *linear system* associated to D is the set $|D|$ of all effective divisors which are equivalent to D , i.e., $|D| = \{E \in \mathbb{Z}^{n+1} : E \geq \vec{0}, E \sim D\}$ and the *degree* of D , written $\deg_R(D)$, is given by $D \cdot R$.

Definition 1.1. For any divisor $D \in \mathbb{Z}^{n+1}$, define the rank of D , denoted by $r(D)$, as follows:

$$r(D) = \min\{\deg_R(E) : |D - E| = \emptyset, E \geq \vec{0}\} - 1.$$

Baker and Norine [2] developed a graph theoretic analogue of the Riemann-Roch formula, originally by studying a certain unrestricted chip-firing game on graphs. Geometrically their result states that for the lattice Λ_G spanned by the rows of the Laplacian of a finite undirected graph G , there exists a *canonical* divisor $K \in \mathbb{Z}^{n+1}$ whose i -th entry is $\deg(v_i) - 2$, of degree $2g - 2$ (where $g = |E(G)| - |V(G)| + 1$) such that for any divisor $D \in \mathbb{Z}^{n+1}$,

$$r(D) - r(K - D) = \deg_{\vec{1}}(D) + 1 - g. \tag{1}$$

Many of their results have since been generalized to a variety of objects including tropical curves, metric graphs and edge weighted graphs [13, 15, 19, 23]. Recently Amini and Manjunath [1] showed that by viewing a the chip-firing game of Baker and Norine geometrically as a walk through the lattice spanned by its Laplacian, a pair of necessary and sufficient Riemann-Roch conditions, equivalent to those of Baker and Norine, could be generalized to all sub-lattices of the lattice $\Lambda_{\vec{1}}$. They refer to these conditions as *uniformity* and *reflection invariance*.

In Section 2, Theorem 2.26 shows that the criteria of Amini and Manjunath [1] naturally extends to any full-dimensional sublattice of Λ_R . Lorenzini [17] gives an alternate Riemann-Roch criteria for such lattices. Our approach differs from his in that we first give a specific rank function (Definition 1.1) and use this to define a pair of necessary and sufficient conditions for a lattice Λ to have the Riemann-Roch property. Lorenzini [17] instead says that such a lattice has the Riemann-Roch property if there exists a *suitable* rank function (§2.1 in [17]), i.e., one which would allow for a Riemann-Roch formula (1) satisfying certain desirable properties. We conclude section 2 with Theorem 2.32 showing that a full-dimensional lattice $\Lambda \subseteq \Lambda_R$ has the Riemann-Roch property if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\vec{1}}$ does, where $\mathcal{R} = \text{diag}(r_0, \dots, r_n)$. This result is later employed when studying the column chip-firing game and when discussing the relationship of chip-firing on arithmetical graphs to the row chip-firing game on associated direct graphs.

Various chip-firing games on graphs have been studied in [3, 4, 5, 12, 14, 21, 22, 24, 28, 29]. Baker and Norine [2] introduced an unrestricted chip-firing game on undirected graphs to prove their Riemann-Roch formula. Their game is as follows: begin with a graph and an integer number of “chips” at each vertex. A vertex either *borrow*s a chip along each of its edges from its neighbors or it *fires*, sending a chip along each of

its edges to its neighbors. The objective of the game is to bring all of the vertices out of debt. In Section 3, we investigate two separate generalizations of the unrestricted chip-firing game on undirected graphs to directed graphs. To understand the two different generalizations of this game to directed graphs we should study how this game relates to the graph Laplacian. The question of whether a configuration D , also called a divisor, can be brought out of debt by some sequence of firings and borrowings is the equivalent to the question of whether $|D| \neq \emptyset$, i.e., $r(D) \geq 0$ for the lattice Λ_G . This is because a sequence of chip-firings corresponds to translation by a lattice point in Λ_G . Let \vec{G} be a directed graph whose adjacency matrix \vec{A} with i, j th entry $\vec{A}_{i,j}$ is the number of edges directed from i to j . Let $\vec{D} = \text{diag}(\deg^+(v_1), \dots, \deg^+(v_n))$ where $\deg^+(v)$ denotes the number edges leaving vertex $v \in V(\vec{G})$. We call the matrix $\vec{Q} = \vec{D} - \vec{A}$ the *Laplacian matrix* of the directed graph \vec{G} . Note that this directed Laplacian is symmetric if and only if it is the Laplacian of a graph with bidirected edges, i.e., an undirected graph. We investigate $r(D)$ and the Riemann-Roch formula for the lattice spanned by the rows of \vec{Q} and the lattice spanned by the columns of \vec{Q} . For both of these lattices, it is equivalent to study certain chip-firing games on \vec{G} . We note that throughout the paper the directed graphs being studied are constrained to be strongly connected, i.e., for any two vertices $i, j \in V(\vec{G})$, there exists a directed path from i to j .

Studying the lattice spanned by the rows of the directed Laplacian is equivalent to studying the row chip-firing game in which if a vertex fires, it will send a chip along each of its outgoing edges. In [2], an important object, called a v_0 -reduced divisor, is introduced. Essentially this is a configuration, where every vertex is out of debt with the possible exception of v_0 and there is no way of "pushing" any money towards v_0 . We generalized this notion of a v_0 reduced divisor to the row chip-firing game on a strongly connected directed graph in Section 3.1.1. In Section 3.1.2, we generalize Dhar's Algorithm, which Baker and Norine used implicitly in [2]. Dhar's algorithm allows one to check whether a divisor whose entries are nonnegative for all vertices other than v_0 is v_0 -reduced and gives, when the divisor is reduced, all of the equivalent v_0 -reduced divisors (for the case of directed graphs, a v_0 -reduced divisor is no longer in general unique). When the divisor is found to not be v_0 -reduced, a firing is obtained, which will bring it "closer" to some v_0 -reduced divisor. In section 4 we present examples which show that lattice spanned by the rows of \vec{Q} may or may not have the Riemann-Roch formula.

We say a directed graph has the strong Riemann-Roch property for directed graphs if it has the Riemann-Roch property and it has a canonical vector K whose i th entry $K(v_i)$ is $\deg^+(v_i) - 2$. We then mention a connection between the sandpile model and the Riemann-Roch property for the row chip-firing game in Section 3.1.3. The directed sandpile model is a constrained version of the row chip-firing game where we restrict our attention to effective divisors. We fire vertices only when they have at least as many chips as their outdegree (so that the divisor remains effective). While many authors require a global sink and ignore the number of chips at this vertex. Because we are studying strongly connected digraphs it is sufficient for our discussion to simply require that a specified vertex v_0 not fire. A divisor D is stable if no vertices may fire and a stable divisor D is recurrent if for every other divisor there exists a way of adding chips to vertices after which the divisor will stabilize to D . We show that for a directed graph \vec{G} , the lattice $\Lambda_{\vec{G}}$ has the strong Riemann-Roch property for directed graphs if and only if for every v_0 -recurrent sandpile configuration D , which is minimal with respect to dominance away from v_0 , there exists $D' = D - ke_0, k \in \mathbb{Z}_{\geq 0}$, which is a continuous extreme divisor. The notion of a continuous extreme divisor is introduced in section 2 and is equivalent to saying that there exist $E_i \in \mathbb{Z}_{\geq 0}$ for $0 \leq i \leq n$ such that $E_i(v_i) = 0$ and $E_i(v_j) > 0$ for $i \neq j$ and $D' \sim E_i$. We note that v_0 -reduced divisors, their connection to v_0 -recurrent sandpile configurations and the generalized Dhar's algorithm were independently discovered by Speer [24] although he was not aware of the connection with Riemann-Roch theory.

Studying the the lattice spanned by the columns is equivalent to studying the column chip-firing game in which if a vertex borrows, it sends a chip along each of its incoming edges and loses a number of chips equal to its outdegree. The number of chips is not conserved, but if we restrict our attention to strongly connected digraphs then we find that there exists a canonical set of currencies, which are integer multiples of some universal currency, with exchange rates so that the game is conservative. In Section 3.2, we explain that the v_0 -reduced divisors for this game are precisely the directed G -parking functions studied in [9]. We show that when studying the column chip-firing game on a strongly connected graph, it is equivalent to

study the row chip-firing game on an associated Eulerian directed graph, that is, a directed graph for which each vertex has the same number of outgoing and incoming edges. We also mention how Dhar's algorithm can be run on a divisor in the column chip-firing game without any serious revision.

We then consider the case of *arithmetical graphs* in Section 4. An arithmetical graph is an undirected multigraph along with a vector $R \in \mathbb{N}^{n+1}$, with $R = (r_0, \dots, r_n)$, where r_i is the weight of vertex v_i subject to the constraint that the sum of the weights of the vertices adjacent to v_i (counting with multiplicity equal the number of edges shared with v_i) you obtain $\delta_i r_i$ for some $\delta_i \in \mathbb{N}$. We define the Laplacian of an arithmetical graph to be the same as for a standard multigraph, but with the i th entry along the diagonal equal to δ_i instead of the degree of v_i . Lorenzini [18] introduced arithmetical graphs as a way of studying the intersection matrices of degenerating curves, which encode some of the discrete data associated with the degeneration. In this paper our interest in arithmetical graphs is derived from the fact that they form a class of vertex weighted graphs whose Laplacian spans an n -dimensional sub-lattice of Λ_R . Indeed, Chung and Langlands [8] introduced a Laplacian matrix for a graph with weights on its vertices, and noted in [17] that if for all $0 \leq i \leq n$ the weight of the vertex v_i is the square of the positive integer r_i , the Laplacian matrix introduced in [8] is the same as the one defined above. The chip-firing game of Baker and Norine extends to arithmetical graphs by assigning to each vertex its own currency, interpreting each vertex's multiplicity as the integer exchange rate between this vertex's currency and the universal chip currency. This is very similar to the notion of currencies employed when studying the column chip-firing game. In doing so we are able to give a combinatorial interpretation of the geometric definitions and statements of Section 2 for arithmetical graphs.

We may obtain from an arithmetical graph (G, R) with Laplacian Q , the Laplacian $\vec{Q} = Q\mathcal{R}$ (where $\mathcal{R} = \text{diag}(r_0, \dots, r_n)$) of a closely related directed graph. In this way we may view arithmetical graphs as a special type of directed graph, particularly since this coordinatewise scaling reduces the chip-firing game for arithmetical graphs to the row chip-firing game for directed graphs and preserves the Riemann-Roch property by Theorem 2.32. In Theorem 4.1 we show that all of the associated directed graphs have the Riemann-Roch property for the column chip-firing game.

Given an arithmetical graph (G, R) we define g_0 by the formula $2g_0 - 2 = \sum_{i=0}^n r_i(\delta_i - 2)$. See [18] for a simple that g_0 is integral and note that g_0 is g for a graph $(G, \vec{1})$. As an application of the tools developed in section 3 we give a combinatorial proof of Proposition 4.2 from [17], which states that $g_{\max} \leq g_0$ and if $g_{\min} = g_{\max} = g_0$ then (G, R) has the Riemann-Roch property (and in particular the associated directed graph has the Riemann-Roch property). The first half of this statement, in the language of chip-firing, says that if there are g_0 chips present in an arithmetical graph then there exists a winning strategy thus generalizing the result of Baker and Norine for arithmetical graphs. The original proof of this result due to Lorenzini was algebro-geometric in nature, employing Riemann-Roch formula for curves.

We conclude with a discussion of some examples of arithmetical graphs, which demonstrate that either, both, or neither of the two Riemann-Roch conditions may be satisfied for an arithmetical graph.

1.1 Basic Notations and Definitions

For any two vectors $x, y \in \mathbb{R}^{n+1}$, let $x \cdot y$ denote the inner product of x and y . For any $x = (x_0, \dots, x_n)^T \in \mathbb{R}^{n+1}$, define $x^+ = (x_0^+, \dots, x_n^+)^T \in \mathbb{R}_+^{n+1}$ and $x^- = (x_0^-, \dots, x_n^-)^T \in \mathbb{R}_-^{n+1}$ to be the *positive part* and *negative part* of x where $x = x^+ + x^-$ and $x_i^+ x_i^- = 0$, for all $0 \leq i \leq n$. Define $\deg_R(x) = R \cdot D$ and call it the *degree* of x . We denote $\deg_R(x^+)$ by $\deg_R^+(x)$ and we call it the *degree plus* of x .

Assume $\vec{0}$ and $\vec{1}$ are the vectors in \mathbb{R}^{n+1} all of whose coordinates are 0 or 1, respectively. For any $x = (x_0, \dots, x_n)^T \in \mathbb{R}^{n+1}$, we say $x \geq \vec{0}$ ($x > \vec{0}$) if and only if for all $0 \leq i \leq n$, $x_i \geq 0$ ($x_i > 0$). We define a *partial order* in \mathbb{R}^{n+1} as follows: for any $x, y \in \mathbb{R}^{n+1}$, we say $x \geq y$ ($x > y$) if and only if $x - y \geq \vec{0}$ ($x - y > \vec{0}$). For any vector $x \in \mathbb{R}^{n+1}$, define $C^+(x) = \{y \in \mathbb{R}^{n+1} : y \geq x\}$ and $C^-(x) = \{y \in \mathbb{R}^{n+1} : x \geq y\}$. We denote the standard basis for \mathbb{R}^{n+1} by $\{e_0, \dots, e_n\}$. Suppose that $R \in \mathbb{N}^{n+1}$ is a vector, and define $H_R = \{x \in \mathbb{R}^{n+1} : R \cdot x = 0\}$. Let $\Lambda_R = H_R \cap \mathbb{Z}^{n+1}$ be the integer lattice in the hyperplane H_R where $R \in \mathbb{N}^{n+1}$. Let $\|\cdot\|$ denote the ℓ^2 -norm, i.e., $\|x\| = \sqrt{x \cdot x}$, for all $x \in \mathbb{R}^{n+1}$.

Let G be graph and let $\{v_0, \dots, v_n\}$ be an ordering of vertices of G . Let $\text{Div}(G)$ be the free Abelian group

on the set of vertices of G . By analogy with the Riemann surface case as noted also in [2], we refer to elements of $\text{Div}(G)$ as *divisors* on G . In the case that the graph G is implied by context, we simply refer to elements of $\text{Div}(G)$ as divisors. Because there is a fixed ordering on vertices of G , we think of an element $\alpha \in \text{Div}(G)$, which is a formal integer linear combinations of vertices of G , as a vector $D = (d_0, \dots, d_n) \in \mathbb{Z}^{n+1}$ where d_i is the coefficient of v_i in α for all $0 \leq i \leq n$. We denote to the i th coordinate of D by $D(v_i)$, for all $0 \leq i \leq n$. We refer to both vectors in \mathbb{Z}^{n+1} and elements of $\text{Div}(G)$ as divisors.

2 Riemann-Roch Theory for Sub-lattices of Λ_R

2.1 Preliminaries

We remark that many of the proofs and statements presented in this section are similar to the ones which appeared in Amini and Manjunath [1]'s work. Essentially, what is being demonstrated is that if one replaces each statement about lattices orthogonal to the all one's vector with the same statement for lattices orthogonal to some fixed positive vector, the proofs will go through without much extra effort. This in itself is not a very strong observation, but it is necessary for proving Theorem 2.26 and Theorem 2.32, which are used several times in the proceeding sections so, for the sake of completeness, we have decided to provide all of the necessary lemmas with proofs.

Throughout this section, R will denote a vector in \mathbb{N}^{n+1} .

Definition 2.1. Let $\Lambda \subseteq \Lambda_R$ be a sub-lattice of rank n . Define

$$\Sigma(\Lambda) = \{D \in \mathbb{Z}^{n+1} : D \not\geq p \text{ for all } p \in \Lambda\},$$

$$\Sigma_{\mathbb{R}}(\Lambda) = \{x \in \mathbb{R}^{n+1} : x \not\geq p \text{ for all } p \in \Lambda\}.$$

Note that the set $\Sigma(\Lambda)$ defined in Definition 2.1 is the negative of the *Sigma region* set defined by Amini and Manjunath [1]. We denote by $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$ the topological closure of the set $\Sigma_{\mathbb{R}}(\Lambda)$ in \mathbb{R}^{n+1} . Let $B(x, r) = \{y \in \mathbb{R}^{n+1} : \|y - x\| \leq r\}$ denote the ball of radius r with center at x . For any set $S \subset \mathbb{R}^{n+1}$, let $\text{int}(S)$ denote the *relative interior* of S .

Lemma 2.2. If $\Lambda \subseteq \Lambda_R$ is a sub-lattice of rank n , then

$$\overline{\Sigma}_{\mathbb{R}}(\Lambda) = \{x \in \mathbb{R}^{n+1} : x \not\geq p, \text{ for all } p \in \Lambda\}.$$

Proof. Suppose $x \in \mathbb{R}^{n+1}$ such that $x \geq p$ for some $p \in \Lambda$. Thus there exists $\delta > 0$ such that for all $y \in B(x, \delta)$, $y \geq p$. Thus $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Now, suppose $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Then there exists $\delta > 0$ and $p \in \Lambda$ such that $x - \frac{\delta}{2}\mathbf{1} \geq p$. Hence $x \geq p$, and this completes the proof of the lemma. \square

Lemma 2.3. If $D \in \mathbb{Z}^{n+1}$ then $D \in \Sigma(\Lambda)$ if and only if $D + \mathbf{1} \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$.

Proof. If $D \notin \Sigma(\Lambda)$, then there exists $p \in \Lambda$ such that $D \geq p$. Hence $D + \mathbf{1} \geq p$ and by Lemma 2.2 $D + \mathbf{1} \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. If $D + \mathbf{1} \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ then Lemma 2.2 implies that $D + \mathbf{1} \geq p$ for some $p \in \Lambda$. Since $D, p \in \mathbb{Z}^{n+1}$, it follows that $D \geq p$ and this implies that $D \notin \Sigma(\Lambda)$. \square

Suppose $R = (r_0, \dots, r_n) \in \mathbb{R}_+^{n+1}$ and $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$. Define $\|x\|_R = \sum_{i=0}^n r_i |x_i|$. It is easy to see that $\|\cdot\|_R$ is a norm on \mathbb{R}^n . For any two points $x, y \in \mathbb{R}^{n+1}$, we define $\text{dist}_R(x, y) = \|x - y\|_R$. One can consider $\|\cdot\|_R$ as a *weighted taxi-cab* distance. For any set $S \subseteq \mathbb{R}^{n+1}$ and $p \in \mathbb{R}^{n+1}$, we define $\text{dist}_R(p, S) = \inf\{\text{dist}_R(p, x) : x \in S\}$. Observe that $r(D) = -1$ if D is not equivalent to any effective divisor and $-1 \leq r(D) \leq \deg_R(D)$.

Lemma 2.4. If $D \in \mathbb{Z}^{n+1}$ is a divisor then

(i) $r(D) = -1$ if and only if $D \in \Sigma(\Lambda)$.

$$(ii) \quad r(D) = \text{dist}_R(D, \Sigma(\Lambda)) - 1 = \min\{\text{dist}_R(D, p) : p \in \Sigma(\Lambda)\} - 1.$$

Proof. (i) For $D \in \mathbb{Z}^{n+1}$, $r(D) = -1$ if and only if for all $p \in \Lambda$, $D - p \not\geq \vec{0}$ if and only if $D \in \Sigma(\Lambda)$.

(ii) Since $\Sigma(\Lambda)$ is a closed set, $\inf\{\text{dist}_R(D, p) : p \in \Sigma(\Lambda)\} = \min\{\text{dist}_R(D, p) : p \in \Sigma(\Lambda)\}$.

$$\begin{aligned} r(D) &= \min\{\deg(E) : |D - E| = \emptyset, E \geq \vec{0}\} - 1 \\ &= \min\{\deg(E) : r(D - E) = -1, E \geq \vec{0}\} - 1 \\ &= \min\{\deg(E) : D - E \in \Sigma(\Lambda), E \geq \vec{0}\} - 1 \\ &= \min\{\deg_R(D - p) : D - p \geq \vec{0}, p \in \Sigma(\Lambda)\} - 1 \\ &= \text{dist}_R(D, \Sigma(\Lambda)) - 1. \end{aligned}$$

Note that the last equality follows from the fact that if $p \in \Sigma(\Lambda)$ and $(D - p)_i < 0$ for some $0 \leq i \leq n$ then $\text{dist}_R(D, p - e_i) \leq \text{dist}_R(D, p)$ and $p - e_i \in \Sigma(\Lambda)$. \square

2.2 Extreme Points of $\Sigma(\Lambda)$ and $\bar{\Sigma}_{\mathbb{R}}(\Lambda)$

Define $H_R^+ = \{x \in \mathbb{R}^{n+1} : x \cdot R \geq 0\}$. For any vector $p \in H_R^+$, define $\Delta_R(p) = H_R \cap C^-(p)$ to be the n -dimensional simplex in the hyperplane H_R . For the definitions of simplex and facet and their properties, we refer the reader to [20, 26]. For simplicity we denote $\Delta_R(R)$ by Δ_R .

It is easy to see that for any $p \in H_R^+$ there exists a unique $\lambda \geq 0$ and $p' \in H_R$ such that $p = p' + \lambda R$. Define the *projection* function $\pi : H_R^+ \rightarrow H_R$ as follows: for any $p \in H_R^+$, define $\pi(p) = p'$. It is also easy to see that $\pi(p) = p - \lambda R$ where $\lambda = (p \cdot R) / \|R\|^2$. We refer to $\pi(p)$ the *projection* of the point p into the hyperplane H_R along the vector R . The following lemma is an immediate consequence of the above definition.

Lemma 2.5. *If $p = (p_0, \dots, p_n) \in H_R^+$ and $p = \pi(p) + \lambda R$, then*

$$(i) \quad \Delta_R(p) = \pi(p) + \lambda \Delta_R.$$

(ii) $F_i = \Delta_R(p) \cap \{x \in \mathbb{R}^n : x_i = p_i\}$ for all $0 \leq i \leq n$, defines all the facets of the simplex $\Delta_R(p)$.

It is easy to see that Δ_R is the simplex in H_R with vertices $b^0, \dots, b^n \in H_R$ whose coordinates are:

$$b_j^i = \begin{cases} -\sum_{k \neq i} \frac{r_k^2}{r_i} & \text{if } i = j \\ \text{otherwise} \end{cases}$$

for all $0 \leq j \leq n$.

Definition 2.6. *For any two points $p, q \in H_R$, define the Δ_R -distance function between p and q as follows:*

$$d_{\Delta_R}(p, q) = \inf\{\lambda \geq 0 : q \in p + \lambda \Delta_R\}.$$

The Δ_R -distance function defined above is a *gauge function* (which is often used in the study of convex bodies). For more on gauge functions and their properties, see [27].

For any point $p \in \Lambda$ define $d_{\Delta_R}(p, \Lambda) = \min\{\lambda \geq 0 : \text{there exists } q \in \Lambda \text{ such that } q \in p + \lambda \Delta_R\}$.

The following remark can be considered as a generalization of Lemma 4.7 in [1], and its proof easily follows from Definition 2.6.

Remark 2.7. Given any two vectors $p, q \in H_R$,

$$d_{\Delta_R}(p, q) = \max_{0 \leq i \leq n} \left\{ \frac{q_i - p_i}{r_i} \right\}.$$

Proof. By Definition 2.6,

$$\begin{aligned} d_{\Delta_R}(p, q) &= \inf\{\lambda \geq 0 : q \in p + \lambda \Delta_R\} = \inf\{\lambda \geq 0 : q \in p + C^-(\lambda R)\} \\ &= \inf\{\lambda \geq 0 : q \leq p + \lambda R\} = \max_{0 \leq i \leq n} \left\{ \frac{q_i - p_i}{r_i} \right\}. \end{aligned}$$

□

Definition 2.8. Define

$$\begin{aligned} \text{Ext}(\Sigma(\Lambda)) &= \{\nu \in \Sigma(\Lambda) : \deg_R(\nu) \geq \deg_R(p), \text{ for all } p \in N(\nu) \cap \Sigma(\Lambda)\}, \\ \text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) &= \{\nu \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) : \exists \delta > 0, \text{ such that } \deg_R(\nu) \geq \deg_R(p), \text{ for all } p \in B(\nu, \delta) \cap \overline{\Sigma}_{\mathbb{R}}(\Lambda)\}, \\ \text{Crit}(\Lambda) &= \{\nu \in H_R : \exists \delta > 0 \text{ such that } d_{\Delta_R}(\nu, \Lambda) \geq d_{\Delta_R}(p, \Lambda), \text{ for all } p \in B(\nu, \delta) \cap H_R\}. \end{aligned}$$

where $N(\nu)$ consists of all points $D \in \mathbb{Z}^{n+1}$ such that $\|D - \nu\|_1 \leq 1$. We call $\text{Ext}(\Sigma(\Lambda))$, $\text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ and $\text{Crit}(\Lambda)$, the set of extreme points or extreme divisors of $\Sigma(\Lambda)$, $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$ and the set of critical points of Λ , respectively.

Lemma 2.9. If $p, q \in H_R^+$, then $p \leq q$ if and only if $\Delta_R(p) \subseteq \Delta_R(q)$. In particular, $p < q$ if and only if $\Delta_R(p) \subsetneq \text{int}(\Delta_R(q))$.

Proof. It is easy to see that $p \leq q$ if and only if $C^-(p) \subseteq C^-(q)$. Now the second part of Lemma 2.5 implies that $C^-(p) \subseteq C^-(q)$ if and only if $(C^-(p) \cap H_R) \subseteq (C^-(q) \cap H_R)$. □

An easy application of Lemma 2.2 is that if $p \in \text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$, then $p \notin \Lambda$. The following theorem characterizes the set of extreme points of $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$.

Theorem 2.10. If $p \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$ then $p \in \text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$ if and only if each facet of the simplex $\Delta_R(p)$ contains a point of Λ in its interior.

Proof. Assume that $p = (p_0, \dots, p_n) \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$. Let F_i , $0 \leq i \leq n$ be the facets of $\Delta_R(p)$. Let $0 \leq i \leq n$ be such that $\text{int}(F_i)$ contains no point of Λ . By Lemma 2.5 (ii), there exists an $\epsilon > 0$ such that $\Delta_R(p + \epsilon e_i)$ does not contain any points of Λ in its interior. Hence Lemma 2.9 and Lemma 2.2 imply that $p + \epsilon e_i \in \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. Since $\deg_R(p) < \deg_R(p + \epsilon e_i)$, the point p is not an extreme point.

Conversely, assume that $p \in \overline{\Sigma}_{\mathbb{R}}(\Lambda) \setminus \Lambda$ is such that the interior of each facet F of $\Delta_R(p)$ contains a point of Λ . We claim that for any $v = (v_0, \dots, v_n) \in \mathbb{R}^{n+1}$, either $\deg_R(p + \epsilon v) \leq \deg_R(p)$ for all $\epsilon \geq 0$, or there exists $\lambda > 0$ such that for all $0 < \epsilon \leq \lambda$, $p + \epsilon v \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$. If $v \leq \vec{0}$, then for all $\epsilon \geq 0$, $\deg_R(p + \epsilon v) \leq \deg_R(p)$. Now, without loss of generality assume that $v_0 > 0$ and $v_1 \leq 0$. Suppose $x \in \text{int}(F)$ where $F = \Delta_R(D) \cap \{y \in \mathbb{R}^n : (y - D) \cdot e_0 = 0\}$. Since $x \in \text{int}(F)$, we can pick $\lambda > 0$ small enough such that for all $0 < \epsilon \leq \lambda$, $x \in \text{int}(\Delta_R(p + \epsilon v))$. Thus Lemma 2.9 and Lemma 2.2 imply that $x \notin \overline{\Sigma}_{\mathbb{R}}(\Lambda)$ for all $0 < \epsilon \leq \lambda$. This completes the proof of the claim. It is easy to see that the proof of the theorem follows from the claim. □

Corollary 2.11. $\text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) \subset \mathbb{Z}^{n+1}$.

Proof. Let $p \in \text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$. Theorem 2.10 shows that the interior of every facet F of $\Delta_R(p)$ contains a point of Λ . Since $\Lambda \subseteq \mathbb{Z}^{n+1}$, the second part of Lemma 2.5 implies that $p \in \mathbb{Z}^{n+1}$. □

Theorem 2.12. A divisor $\nu \in \text{Ext}(\Sigma(\Lambda))$ if and only if $\nu + \vec{1} \in \text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda))$.

Proof. Corollary 2.11 implies that $\text{Ext}(\overline{\Sigma}_{\mathbb{R}}(\Lambda)) \subseteq \mathbb{Z}^{n+1}$. The theorem immediately follows from Lemma 2.3. □

The set of critical points of Λ ($\text{Crit}(\Lambda)$ in Definition 2.8) is the set of local maxima of the function $d_{\Delta_R}(\cdot, \Lambda)$. The following theorem characterizes critical points of Λ in terms of extreme points of $\overline{\Sigma}_{\mathbb{R}}(\Lambda)$.

Theorem 2.13. For $p \in H_R$, let $\lambda = d_{\Delta_R}(p, \Lambda)$ and $p' = p + \lambda R$. Then $p' \in \text{Ext}(\overline{\Sigma}_R(\Lambda))$ if and only if $p \in \text{Crit}(\Lambda)$.

Proof. If $p' \in \text{Ext}(\overline{\Sigma}_R(\Lambda))$ then by Theorem 2.10 each facet of the simplex $\Delta_R(p + \lambda R) = p + \lambda \Delta_R$ contains a point of Λ in its interior. This shows that $p \in \text{Crit}(\Lambda)$.

Conversely, assume that $p \in \text{Crit}(\Lambda)$ and $p' \notin \text{Ext}(\overline{\Sigma}_R(\Lambda))$. As the proof of Theorem 2.10 shows, there exist $0 \leq i \leq n$ and $\delta > 0$ such that for all $0 < \epsilon \leq \delta$, $p'_\epsilon = p' + \epsilon e_i \in \overline{\Sigma}_R(\Lambda)$. For each $0 < \epsilon \leq \delta$, let $p_\epsilon = \pi(p'_\epsilon)$ to be the projection of p'_ϵ along R into H_R . Lemma 2.15 implies that $d_{\Delta_R}(p_\epsilon, \Lambda) = \left(\frac{p'_\epsilon \cdot R}{\|R\|^2}\right)$. Since $p'_\epsilon \cdot R > p' \cdot R$, we conclude that $d_{\Delta_R}(p_\epsilon, \Lambda) > d_{\Delta_R}(p, \Lambda)$, a contradiction. \square

Corollary 2.14. Let $\varphi : \text{Ext}(\Sigma(\Lambda)) \rightarrow \text{Crit}(\Lambda)$ be as follows: For any $\nu \in \text{Ext}(\Sigma(\Lambda))$, $\varphi(\nu) = \pi(\nu + \vec{1})$. Then φ is a bijection.

Proof. This follows from Theorems 2.13 and 2.12. \square

Lemma 2.15. Let $p \in H_R$, $\lambda = d_{\Delta_R}(p, \Lambda)$ and $\lambda' = \max\{t \geq 0 : p + tR \in \overline{\Sigma}_R(\Lambda)\}$. Then $\lambda = \lambda'$.

Proof. First note that since $p \in \overline{\Sigma}_R(\Lambda)$ and $\overline{\Sigma}_R(\Lambda)$ is a closed set, $\max\{t \geq 0 : p + tR \in \overline{\Sigma}_R(\Lambda)\}$ is well-defined. The first part of Lemma 2.5 implies that $p + t\Delta_R = \Delta_R(p + tR)$. Now, for all $0 \leq t \leq \lambda$, by applying Lemma 2.2 and Lemma 2.9, we conclude that $p + tR \in \overline{\Sigma}_R(\Lambda)$. So $\lambda' \geq \lambda$. Conversely, suppose $t \geq 0$ is such that $\Lambda \cap (p + t\Delta_R) \neq \emptyset$. Lemma 2.2 and Lemma 2.9 imply that $p + tR \in \overline{\Sigma}_R(\Lambda)$ if and only if $\Lambda \cap \text{int}(p + t\Delta_R) = \emptyset$. This shows that $\lambda' \leq \lambda$, completing the proof of the lemma. \square

Lemma 2.16. There exists a constant C depending only on the lattice Λ and the vector R such that for any point $p \in \Sigma(\Lambda)$, we have:

- (i) $\deg_R(p) \leq C$,
- (ii) there exists some $\nu \in \text{Ext}(\Lambda)$ such that $p \leq \nu$.

Proof. (i): First, we claim that there exists c such that for all $p \in H_R$, $d_{\Delta_R}(p, \Lambda) \leq c$. We start by noting that there exists a constant K depending only on R such that $d_{\Delta_R}(p, q) \leq K \cdot \|p - q\|$. This follows immediately by letting the constant K be the largest radius of a sphere in H_R with center at the origin contained in Δ_R .

Let $\{l_0, \dots, l_{n-1}\}$ be a set of generators of Λ , and let P be the parallelotope generated by l_0, \dots, l_{n-1} . Because the Δ_R -distance function is invariant under translation by lattice points, it is sufficient to prove the claim for all $p \in P$. By letting c be K times the maximum ℓ^2 -distance from a point in P to the vertices of P (diameter of P by ℓ^2 -norm), the claim is proved.

To prove the first part, it is enough to show that for all $p \in H_R^+ \cap \Sigma(\Lambda)$, $\deg_R(p) \leq C$. Let $p' = \pi(p)$, $\lambda \geq 0$ be such that $p = p' + \lambda R$. Lemma 2.9 implies that $p \in \Sigma(\Lambda)$ if and only if $\Delta_R(p)$ contains no points of Λ . Lemma 2.15 and Theorem 2.12 imply that $\lambda \leq \text{dist}_{\Delta_R}(p, \Lambda)$, so $\lambda \leq c$. Therefore, $\deg_R(p) = \lambda \|R\|^2 \leq c \|R\|^2$. This shows that $C \leq c \|R\|^2$, which completes the proof of the first part.

(ii): Let $p \in \Sigma(\Lambda)$. The first part shows that the degrees of points in $\text{Ext}(\Lambda)$ are bounded above by C . Therefore $C^+(p) \cap \Sigma(\Lambda)$ is a finite set. This immediately shows that there exists $\nu \in \text{Ext}(\Lambda)$ such that $p \leq \nu$. To be more precise, one can find an extreme point $\nu \in \text{Ext}(\Lambda)$ greedily by starting at point p and walking in positive directions as much as possible. \square

Lemma 2.17. For any divisor $D \in \mathbb{Z}^{n+1}$, $r(D) = \min\{\deg_R^+(D - \nu) : \nu \in \text{Ext}(\Lambda)\} - 1$.

Proof. First we show that $\min\{\deg_R^+(D - \nu) : \nu \in \text{Ext}(\Lambda)\} \leq r(D) + 1$. Let $E \geq \vec{0}$ with $\deg_R(E) = r(D) + 1$ be such that $D - E \in \Sigma(\Lambda)$, where the existence of E guaranteed by Lemma 2.4. By Lemma 2.16, there exists $\nu \in \Sigma(\Lambda)$ such that $\nu \geq D - E$. Let $E' = \nu - (D - E)$. We claim that $E' \cdot E = 0$. Suppose not and assume there exists $0 \leq i \leq n$ such that $E_i, E'_i \geq 1$. Note that $D - (E - e_i) \in \Sigma(\Lambda)$ as $\nu \geq D - (E - e_i)$, but $\deg_R(E - e_i) < \deg_R(E) = r(D) + 1$, a contradiction. This gives that $\deg_R^+(D - \nu) = \deg_R^+(E - E') = \deg(E) = r(D) + 1$.

For proving the reverse inequality, let $\nu \in \text{Ext}(\Lambda)$ be such that $\deg^+(D - \nu)$ is minimum. Because $\nu \geq \nu + (D - \nu)^- = D - (D - \nu)^+$, it follows that $D - (D - \nu)^+ \in \Sigma(\Lambda)$. Hence Lemma 2.4 implies that $r(D) \leq \min\{\deg_R^+(D - \nu) : \nu \in \text{Ext}(\Lambda)\} - 1$, which completes the proof. \square

2.3 Riemann-Roch Theorem for Uniform and Reflection Invariant Sub-lattices of Λ_R

Definition 2.18. Let Λ be a sub-lattice of Λ_R of rank n , and $Ext(\Sigma(\Lambda))$ be the set of extreme points of $\Sigma(\Lambda)$. Define

$$\begin{aligned} g_{\min} &= \min\{\deg_R(\nu) : \nu \in Ext(\Sigma(\Lambda))\} + 1, \\ g_{\max} &= \max\{\deg_R(\nu) : \nu \in Ext(\Sigma(\Lambda))\} + 1. \end{aligned}$$

We say the lattice Λ is uniform if $g_{\min} = g_{\max}$.

Definition 2.19. Let Λ be a sub-lattice of Λ_R of rank n . We say Λ is reflection invariant if $-Crit(\Lambda)$ is a translate of $Crit(\Lambda)$, i.e., if there exists $v \in \mathbb{R}^{n+1}$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$.

Definition 2.20. Let Λ be a sub-lattice of dimension n of Λ_R . We say a divisor $K \in \mathbb{Z}^{n+1}$ is a canonical divisor of Λ , or equivalently Λ has a canonical divisor K , if for all divisors $D \in \mathbb{Z}^{n+1}$,

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq r(D) - r(K - D) \leq \deg_R(D) - g_{\min} + 1.$$

Lemma 2.21. Suppose $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ is a bijection between sets, and $f : \mathcal{A} \rightarrow \mathbb{Z}$ and $f' : \mathcal{A}' \rightarrow \mathbb{Z}$ are functions whose values are bounded from below. If there exist constants $c_1, c_2 \in \mathbb{Z}$ such that for all $a \in \mathcal{A}$,

$$c_1 \leq f(a) - f'(\phi(a)) \leq c_2,$$

then

$$c_1 \leq \min_{a \in \mathcal{A}} f(a) - \min_{a' \in \mathcal{A}'} f'(a') \leq c_2.$$

Proof. Since f and f' are integer valued functions whose values are bounded from below, there exists $x \in \mathcal{A}$ and $y \in \mathcal{A}'$ such that $f(x) = \min_{a \in \mathcal{A}} f(a)$ and $f'(y) = \min_{a' \in \mathcal{A}'} f'(a')$. The choice of x and y implies that $f(x) - f'(y) \leq f(\phi^{-1}(y)) - f'(y) \leq c_2$, and $f(x) - f'(y) \geq f(x) - f'(\phi(x)) \geq c_1$. Hence $c_1 \leq f(x) - f'(y) \leq c_2$, as desired. \square

Theorem 2.22. Let Λ be a reflection invariant sub-lattice of Λ_R of rank n . Then Λ has a canonical divisor, i.e. there exists a divisor K such that for all $D \in \mathbb{Z}^{n+1}$,

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq r(D) - r(K - D) \leq \deg_R(D) - g_{\min} + 1.$$

Proof. First we construct the canonical divisor K and then we show it has the desired property. Since Λ is reflection invariant, there exists a vector $v \in \mathbb{R}^{n+1}$ such that $-Crit(\Lambda) = Crit(\Lambda) + v$. Therefore there exists a bijection function η from $Crit(\Lambda)$ to itself such that $\eta(c) + c = v$. Let $\varphi : Ext(\Sigma(\Lambda)) \rightarrow Crit(\Lambda)$ be the bijection described in Corollary 2.14. Define the bijection ϕ from $Ext(\Sigma(\Lambda))$ to itself so that for all $\nu \in Ext(\Sigma(\Lambda))$, $\phi(\nu) = \varphi^{-1}\eta\varphi(\nu)$. Since for all $\nu \in Ext(\Sigma(\Lambda))$, $\deg_R(\nu + \phi(\nu)) \leq 2g_{\max}$, there exists $\nu_0 \in Ext(\Sigma(\Lambda))$ such that $\deg_R(\nu_0 + \phi(\nu_0))$ is as large as possible. Let the canonical divisor K be $\nu_0 + \phi(\nu_0)$.

For any $\nu \in Ext(\Sigma(\Lambda))$, let $c = \varphi(\nu)$; then we have:

$$\phi(\nu) + \nu = \phi(\varphi^{-1}(c)) + \varphi^{-1}(c) = \varphi^{-1}\eta(c) + \varphi^{-1}(c) = \lambda R + v - 2 \times \vec{1},$$

where $\lambda \in \mathbb{R}$ is a constant depends on ν (or equivalently c). Hence, the choice of K implies that for any $\nu \in Ext(\Sigma(\Lambda))$, there exists $E_\nu \in \mathbb{R}_+^{n+1}$ such that $\phi(\nu) + \nu + E_\nu = K$. Therefore, for all divisor $D \in \mathbb{Z}^{n+1}$ and $\nu \in Ext(\Sigma(\Lambda))$ we have:

$$\begin{aligned} \deg_R^+(D - \nu) - \deg_R^+(K - D - \phi(\nu)) &= \deg_R^+(D - \nu) - \deg_R^+(\phi(\nu) + \nu + E_\nu - D - \phi(\nu)) \\ &= \deg_R^+(D - \nu) - \deg_R^+(\nu + E_\nu - D) \\ &\leq \deg_R^+(D - \nu) - \deg_R^+(\nu - D) \\ &= \deg_R(D) - \deg_R(\nu) \\ &\leq \deg_R(D) - g_{\min} + 1. \end{aligned}$$

Note that for all $\nu \in \text{Ext}(\Sigma(\Lambda))$, $E_\nu = K - (\nu + \phi(\nu)) \leq 2g_{\max} - 2g_{\min}$. Hence,

$$\begin{aligned}
\deg_R^+(D - \nu) - \deg_R^+(K - D - \phi(\nu)) &= \deg_R^+(D - \nu) - \deg_R^+(\phi(\nu) + \nu + E_\nu - D - \phi(\nu)) \\
&= \deg_R^+(D - \nu) - \deg_R^+(\nu + E_\nu - D) \\
&\geq \deg_R^+(D - \nu) - \deg_R^+(\nu - D) - 2(g_{\max} - g_{\min}) \\
&= \deg_R(D) - \deg_R(\nu) - 2g_{\max} + 2g_{\min} \\
&\geq \deg_R(D) - 3g_{\max} + 2g_{\min} + 1.
\end{aligned}$$

Therefore for all $D \in \mathbb{Z}^{n+1}$ and all $\nu \in \text{Ext}(\Sigma(\Lambda))$,

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq \deg_R^+(D - \nu) - \deg_R^+(K - D - \phi(\nu)) \leq \deg_R(D) - g_{\min} + 1.$$

For a fixed $D \in \mathbb{Z}^{n+1}$, $\deg_R(D) - 3g_{\max} + 2g_{\min} + 1$ and $\deg_R(D) - g_{\min} + 1$ are constant integers, $\deg_R^+(D - \nu)$ and $\deg_R^+(K - D - \phi(\nu))$ are integer value functions bounded from below by zero, and ϕ is a bijection from $\text{Ext}(\Sigma(\Lambda))$ to itself, hence Lemma 2.21 implies that

$$\deg_R(D) - 3g_{\max} + 2g_{\min} + 1 \leq \min_{\nu \in \text{Ext}(\Sigma(\Lambda))} \deg_R^+(D - \nu) - \min_{\nu \in \text{Ext}(\Sigma(\Lambda))} \deg_R^+(K - D - \nu) \leq \deg_R(D) - g_{\min} + 1.$$

The assertion of the theorem now follows from Lemma 2.17. \square

Definition 2.23. Let Λ be a uniform sub-lattice of dimension n of Λ_R . We say Λ has the Riemann-Roch property if there exists a divisor K with degree $2g - 2$, where $g = g_{\min} = g_{\max}$, such that for all divisor $D \in \mathbb{Z}^{n+1}$:

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Theorem 2.24. Let Λ be a uniform sub-lattice of dimension n of Λ_R . Then Λ is reflection invariant if and only if Λ has the Riemann-Roch property.

Proof. Assume Λ is reflection invariant and let K be the canonical divisor obtained in the proof of Theorem 2.22. By applying Theorem 2.22, it's enough to show that $\deg(K) = 2g - 2$. The construction of K shows that $K = \nu + \phi(\nu)$, where ϕ is the bijection obtained in proof of Theorem 2.22. Since Λ is uniform, $g_{\min} = g_{\max} = g$. Hence $\deg_R(\nu) = \deg_R(\phi(\nu)) = g - 1$ and this implies that $\deg_R(K) = 2g - 2$.

Now, assume that Λ has the Riemann property. Assume ν is an extreme divisor of $\Sigma(\Lambda)$, so the first part of Lemma 2.4 implies that $r(\nu) = -1$. Since Λ is uniform $\deg_R(\nu) = g - 1$ and this shows that $r(K - \nu) = r(\nu) = -1$. By Lemma 2.4, $K - \nu \in \Sigma(\Lambda)$, and is hence an extreme divisor of $\Sigma(\Lambda)$. Hence the function ψ defined as $\psi(-\nu) = K - \nu$, for all $\nu \in \text{Ext}(\Lambda)$ is a bijection from $\text{Ext}(\Lambda)$ to itself. If φ is the function defined in Corollary 2.14, the function $\varphi \circ \psi \circ \varphi^{-1}$ is a bijection from $\text{Crit}(\Lambda)$ to itself. It is easy to see that for any $p \in \text{Crit}(\Lambda)$, $\varphi(\psi(\varphi^{-1}(p))) = -p + \pi(K) + 2\pi(\bar{\mathbf{I}})$, and by picking $v = -\pi(K) - 2\pi(\bar{\mathbf{I}})$, we have $-\text{Crit}(\Lambda) = \text{Crit}(\Lambda) + v$. \square

Definition 2.25. We say a sub-lattice Λ of Λ_R has the Riemann-Roch formula if there exists an integer $m \in \mathbb{Z}$ and a divisor K of degree $2m - 2$ such that for all $D \in \mathbb{Z}^{n+1}$:

$$r(D) - r(K - D) = \deg_R(D) - m + 1.$$

Theorem 2.26. Let Λ be a sub-lattice of dimension n of Λ_R . Then Λ has a Riemann-Roch formula if and only if Λ is uniform and reflection invariant, in particular Λ has the Riemann-Roch property.

Proof. If Λ is uniform and reflection invariant, then Theorem 2.24 implies that Λ has Riemann-Roch property and therefore Λ has the Riemann-Roch formula with $m = g_{\max}$.

For proving the other direction it is enough by Theorem 2.24 to show that Λ is uniform and $m = g_{\max}$. First, we show that $m = g_{\max}$. Let D be a divisor with $\deg_R(D) \geq m$. The Riemann-Roch formula implies that $r(D) - r(K - D) \geq 1$ and since $r(K - D) \geq -1$, we have $r(D) \geq 0$. It follows that $g_{\max} \leq m$.

We know that for any divisor $D \in \mathbb{Z}^{n+1}$, if the degree of D is more than $g_{\max} - 1$ then the divisor is effective, so $\deg_R(D) - r(D) \leq g_{\max}$. On the other hand, if $\deg_R(D) > 2m - 2$, then $\deg_R(K - D) < 0$, therefore $r(K - D) = -1$. The Riemann-Roch formula implies that $\deg(D) - r(D) = m$. Therefore, $m \leq g_{\max}$. This shows that $m = g_{\max}$.

To prove uniformity, let $\nu \in \text{Ext}(\Sigma(\Lambda))$ and $\deg_R(\nu) < g_{\max} - 1$. Since $\deg_R(K) = 2g_{\max} - 2$, $\deg_R(K - \nu) \geq g_{\max}$, so $K - \nu \notin \Sigma(\Lambda)$, and by Lemma 2.4 is equivalent to an effective divisor. The Riemann-Roch formula implies that $r(K - \nu) = g_{\max} - \deg(\nu) - 2$, so there exists an effective divisor E of degree $g_{\max} - \deg(\nu) - 1 > 0$ such that $|K - \nu - E| = \emptyset$. We claim that $\nu + E$ is not equivalent to an effective divisor. The Riemann-Roch formula implies that $r(\nu + E) - r(K - \nu - E) = \deg_R(\nu + E) - g_{\max} + 1 = 0$ and therefore $r(\nu + E) = -1$. By Lemma 2.4, $\nu + E \in \Sigma(\Lambda)$, contradicting the fact that $\nu \in \text{Ext}(\Sigma(\Lambda))$. \square

2.4 Riemann-Roch Theorem for sub-lattice of Λ_R and $\Lambda_{\bar{\Gamma}}$

Let $R = (r_0, \dots, r_n) \in \mathbb{N}^{n+1}$ and $\mathcal{R} = \text{diag}(r_0, \dots, r_n)$ be a matrix mapping Λ_R to $\Lambda_{\bar{\Gamma}}$. To be more precise, for any $p \in \Lambda_R$ the image of p is $\mathcal{R}p$. For any set $S \subseteq \mathbb{R}^{n+1}$, let $\mathcal{R}S$ denote the set $\{\mathcal{R}p : p \in S\}$. It is easy to see that if $\Lambda \subseteq \Lambda_R$ is a sub-lattice of dimension n then $\mathcal{R}\Lambda$ is a sub-lattice of $\Lambda_{\bar{\Gamma}}$ of dimension n .

Lemma 2.27. *Let Λ be a sub-lattice of dimension n of Λ_R . Then $\mathcal{R}\Sigma(\Lambda) = \Sigma(\mathcal{R}\Lambda)$.*

The proof of above lemma follows easily from Definition 2.1 and the fact that \mathcal{R} is an invertible matrix with positive diagonal entries.

Lemma 2.28. *Let Λ be a sub-lattice of dimension n of Λ_R . Then $\mathcal{R}\text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda)) = \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\mathcal{R}\Lambda))$.*

Proof. Let $\nu \in \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda))$ so that there exists some $\delta > 0$ such that for all $p \in B(\nu, \delta) \cap \bar{\Sigma}_{\mathbb{R}}(\Lambda)$, $\deg_R(\nu) \geq \deg_R(p)$. Let $\delta' = \delta$. It is easy to see that if $q \in B(\mathcal{R}\nu, \delta')$, we have $\mathcal{R}^{-1}q \in B(\nu, \delta)$. Hence $\deg_R(\mathcal{R}^{-1}q) \leq \deg_R(\nu)$ and therefore $\deg_{\bar{\Gamma}}(q) \leq \deg_{\bar{\Gamma}}(\mathcal{R}\nu)$. Here we have used the fact that for any $D \in \mathbb{Z}^{n+1}$, $\deg_R(D) = \deg_{\bar{\Gamma}}(\mathcal{R}D)$ and Lemma 2.27. This proves that $\mathcal{R}\text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda)) \subseteq \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\mathcal{R}\Lambda))$. The other direction is proved similarly. \square

The following corollary immediately follows from Lemma 2.28 and Theorem 2.12.

Corollary 2.29. *Let Λ be a sub-lattice of dimension n of Λ_R . Then Λ is uniform if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\bar{\Gamma}}$ is uniform.*

Lemma 2.30. *Let Λ be a uniform sub-lattice of dimension n of Λ_R . Then Λ is reflection invariant if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\bar{\Gamma}}$ is reflection invariant.*

Proof. First suppose Λ is reflection invariant. Then there exists a vector $v \in \mathbb{R}^{n+1}$ such that $-\text{Crit}(\Lambda) = \text{Crit}(\Lambda) + v$. By applying Lemma 2.28 and Theorem 2.13, let $\mathcal{R}\nu - \bar{\mathbf{1}} - \deg_{\bar{\Gamma}}(\mathcal{R}\nu - \bar{\mathbf{1}})\bar{\mathbf{1}}$ be an arbitrary point of $\text{Crit}(\mathcal{R}\Lambda)$ where ν is an arbitrary point of $\text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda))$. Now, by applying Theorem 2.13,

$$\nu - \bar{\mathbf{1}} - \deg_R(\nu - \bar{\mathbf{1}})R \in \text{Crit}(\Lambda).$$

Since Λ is reflection invariant, there exists $\nu' \in \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda))$ such that

$$-\nu + \bar{\mathbf{1}} + \deg_R(\nu - \bar{\mathbf{1}})R = \nu' - \bar{\mathbf{1}} - \deg_R(\nu' - \bar{\mathbf{1}})R + v,$$

therefore

$$-\mathcal{R}\nu + \mathcal{R}\bar{\mathbf{1}} + \deg_R(\nu - \bar{\mathbf{1}})\mathcal{R}R = \mathcal{R}\nu' - \mathcal{R}\bar{\mathbf{1}} - \deg_R(\nu' - \bar{\mathbf{1}})\mathcal{R}R + \mathcal{R}v.$$

Since Λ is uniform $\deg_R(\nu - \bar{\mathbf{1}})$ is a constant independent from the choice of $\nu \in \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda))$. Hence, $\mathcal{R}\nu - \mathcal{R}\nu' = u$ where u is constant vector in \mathbb{R}^{n+1} which does not depend on ν or ν' . Since $\mathcal{R}\Lambda$ is uniform, $\deg_{\bar{\Gamma}}(\mathcal{R}\nu - \bar{\mathbf{1}})$ is a constant independent from the choice of $\nu \in \text{Ext}(\bar{\Sigma}_{\mathbb{R}}(\Lambda))$. This shows that

$$\mathcal{R}\nu - \mathcal{R}\nu' = u + 2\deg_{\bar{\Gamma}}(\mathcal{R}\nu - \bar{\mathbf{1}}) + 2 \times \bar{\mathbf{1}}.$$

Hence $\mathcal{R}\Lambda$ is reflection invariant. The other direction is proved similarly. \square

Recall the definition of the canonical vector (Definition 2.20) and the argument in the proof of Lemma 2.22 in constructing a canonical vector for a reflection invariant sublattice of Λ_R . So we can consider the following corollary as a consequence of Theorem 2.12, Lemma 2.28, and Lemma 2.30.

Corollary 2.31. *Let Λ be a reflection invariant sub-lattice of dimension n of Λ_R . If K is a canonical vector of $\mathcal{R}\Lambda$ then $\mathcal{R}^{-1}(K + 2 \times \vec{1}) - 2 \times \vec{1}$ is a canonical vector of Λ .*

The following theorem immediately follows from Theorem 2.26, Corollary 2.29 and Lemma 2.30.

Theorem 2.32. *Let Λ be a uniform sub-lattice of dimension n of Λ_R . Then Λ has the Riemann-Roch property if and only if $\mathcal{R}\Lambda \subseteq \Lambda_{\vec{1}}$ has the Riemann-Roch property.*

3 Chip-Firing Game on Directed Graphs

3.1 Row Chip-Firing Game, The Sandpile Model and Riemann-Roch Theory

Let \vec{G} be a directed graph with vertex set $\{v_0, \dots, v_n\}$ and adjacency matrix \vec{A} whose entry $\vec{A}_{i,j}$ for $0 \leq i, j \leq n$ is the number of edges directed from v_i to v_j . Let $\vec{D} = \text{diag}(\deg^+(v_0), \dots, \deg^+(v_n))$ where $\deg^+(v)$ denotes the number edges leaving vertex $v \in V(\vec{G})$. We call the matrix $\vec{Q} = \vec{D} - \vec{A}$ the *Laplacian matrix* of the directed graph \vec{G} . We define $\Lambda_{\vec{G}}$ to be the lattice spanned by the rows of \vec{Q} .

In this section we study the following row chip-firing game on vertices of \vec{G} . Begin with $D \in \mathbb{Z}^{n+1}$, which we call a configuration or a divisor, whose i th entry $D(v_i)$ is the number of chips at vertex v_i . In each *move* of the game either a vertex *borrow*s or *fires*. We say a vertex *fires* if it sends a chip along each of its outgoing edges to its neighbors and *borrow*s if it receives a chip along each of its incoming edges from its neighbors. We say that a vertex is in *debt* if the number of chips at that vertex is negative. The objective of the game is to bring every vertex out of debt by some sequence of moves. Note that the game is “commutative” in the sense that the order of firings and borrowings does not effect the final configuration. For $f \in \mathbb{Z}^{n+1}$, we may interpret the divisor $D' = D - \vec{Q}^T f$ as the divisor obtained from D by a sequence of moves in which the vertex v_i fires $f(v_i)$ times if $f(v_i) \geq 0$ and it borrows $f(v_i)$ times if $f(v_i) \leq 0$. We refer to f as a *firing strategy*. Note that both firing strategies and divisors are vectors in \mathbb{Z}^{n+1} . We say a configuration is a *winning configuration* if all of the vertices are out of debt. We call a sequence of moves which achieves a winning configuration a *winning strategy*. The question of whether a winning strategy exists is equivalent to the question of whether there exists a firing strategy $f \in \mathbb{Z}^{n+1}$ and an effective divisor $E \in \mathbb{Z}_{\geq 0}^{n+1}$ such that $E = D + \vec{Q}^T f$, i.e., $D - E \in \Lambda_{\vec{G}}$, $|D| \neq \emptyset$ or $r(D) \geq 0$. In what follows we will restrict our attention to strongly connected directed graphs. The main motivation for this consideration is given in the following lemma which, interpreted combinatorially, characterizes strongly connected digraphs in terms of which firings leave a divisor unaffected.

Lemma 3.1. *A directed graph \vec{G} is strongly connected if and only if there exists a vector $R \in \mathbb{N}^{n+1}$, unique up to multiplication by a real constant, such that $\vec{Q}^T R = 0$.*

Proof. Suppose \vec{G} is strongly connected. For the sake of contradiction suppose there exists $R \not\geq 0$ such that $\vec{Q}^T R = 0$. Let V^+ be the set of vertices of \vec{G} such that $R(v) > 0$ for all $v \in V^+$. Let $D = \vec{Q}^T R$. Since the net amount of chips leaving V^+ is positive, there must exist some $v \in V^+$ such that $D(v) < 0$, a contradiction. Now assume there exist two linearly independent firing strategies R_1 and R_2 then it is easy to see that there exists a linear combination of R_1 and R_2 , say R , such that $R \not\geq 0$. This proves the uniqueness. Note that we can take R to be an integral vector.

Conversely, suppose \vec{G} is not strongly connected. Let V_1, \dots, V_t be the decomposition of vertices of \vec{G} into maximal strongly connected components. Without loss of generality, let V_1 be a set of vertices such that there exists no edges from u to v where $u \in V_i$, $2 \leq i \leq t$ and $v \in V_1$. As above there exists $v \in V_1$ such that $\vec{Q}^T R(v) < 0$, a contradiction. \square

3.1.1 Reduced Divisors

Let $f, f' \in \mathbb{Z}^{n+1}$ be firing strategies. We define an equivalence relation \approx on \mathbb{Z}^{n+1} by declaring $f \approx f'$ if $\vec{Q}^T(f - f') = \vec{0}$. For any set $S \subseteq V(\vec{G})$, the *characteristic vector* of S , denoted by χ_S , is the vector $\sum_{v_i \in S} e_i$. We say a vector $f \in \mathbb{Z}^{n+1}$ is a *natural* firing strategy if $f \leq R$, and $f \not\leq \vec{0}$. We say a nonzero vector $f \in \mathbb{Z}^{n+1}$ is a *valid* firing strategy with respect to v_0 if $f(v_0) = 0$, and $\vec{0} \leq f \leq R$. The following lemma is an immediate consequence of Lemma 3.1.

Lemma 3.2. *Let $f \in \mathbb{Z}^{n+1}$ be a nonzero firing strategy then there exists a unique $f' \in \mathbb{Z}^{n+1}$ such that $f \approx f'$ and f' is a natural firing strategy.*

Definition 3.3. *Let \vec{G} be a directed graph. We call a divisor D v_0 -reduced if the following two conditions hold:*

- (i) *for all $v \in V(\vec{G}) \setminus \{v_0\}$, $D(v) \geq 0$,*
- (ii) *for every valid firing f with respect to v_0 , there exists a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f)(v) < 0$.*

The following remark immediately follows from Definition 3.3.

Remark 3.4. If $D' \sim D$ is a v_0 -reduced divisor then for all $k \in \mathbb{Z}$, $D' + k\chi_{\{v_0\}}$ is a v_0 -reduced divisor and $D' + k\chi_{\{v_0\}} \sim D + k\chi_{\{v_0\}}$.

Lemma 3.5. *Let D be a v_0 -reduced divisor and let f be a firing strategy such that $f(v_0) \leq 0$ and $f(v) > 0$ for some vertex $v \in V(\vec{G}) \setminus \{v_0\}$. Then there exists $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f)(v) < 0$.*

Proof. Lemma 3.2 implies that there exists a natural firing strategy $f' \approx f$ with $f'(v_0) \leq f(v_0) = 0$. Suppose f^+ and f^- are the positive and negative part of f' . It is easy to see that f^+ is a valid firing strategy with respect to v_0 . Hence there exists a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ such that $(D - \vec{Q}^T f^+)(v) < 0$. Therefore,

$$(D - \vec{Q}^T f)(v) = (D - \vec{Q}^T f')(v) = (D - \vec{Q}^T f^+ - \vec{Q}^T f^-)(v) \leq (D - \vec{Q}^T f^+)(v) < 0.$$

□

Lemma 3.6. *Let \vec{G} be a directed graph and let D be a divisor. Then there exists a divisor $D' \sim D$ such that D' is v_0 -reduced.*

Proof. The proof that we present here is similar to the proof presented by Baker and Norine [2](§3.1). The process of obtaining a v_0 -reduced divisor $D' \sim D$ has two steps: first we bring every $v \in V(\vec{G}) \setminus \{v_0\}$ out of debt, so that it satisfies the first condition of Definition 3.3, and then we “reduce” the divisor with respect to v_0 , in order to satisfy the second condition of Definition 3.3. For performing the first step, define $d(v)$, for all $v \in V(\vec{G}) \setminus \{v_0\}$, to be the length of the shortest directed path from v_0 to v . Let $d = \max_{v \in V(\vec{G}) \setminus \{v_0\}} d(v)$. For all $1 \leq i \leq d$, define $A_i = \{v \in V(\vec{G}) : d(v) = i\}$. Now we bring the A_i ’s out of debt consecutively, starting at A_d . We recursively define sequences of integers b_i and divisors D_i as follows. Let $b_d = \max(\{-D(v) : v \in A_d, D(v) \leq 0\} \cup \{0\})$. Define $D_d = D - \vec{Q}^T f_d$ where f_d is the all zero vector except $f_d(v_j) = b_d$ if $v_j \notin A_d$. It is easy to see that $D_d(v_j) \geq 0$ for all $v_j \in A_d$. Now suppose $1 \leq i \leq d-1$, and define $b_i = \max(\{-D(v) : v \in A_i, D_{i+1}(v) \leq 0\} \cup \{0\})$. Define $D_i = D_{i+1} - \vec{Q}^T f_i$ where f_i is the all zero vector except $f_i(v_j) = b_i$ if $v_j \notin \bigcup_{k=i}^d A_k$. It is easy to see that $D_i(v_j) \geq 0$ for all $v_j \in A_i$ and $D_i(v_j) = D_{i+1}(v_j)$ for all $v_j \in \bigcup_{k=i+1}^d A_k$. Since d is a finite number and the b_i ’s are bounded, the above procedure terminates. It is easy to verify that $D_1 \sim D$ is a divisor such that no vertex other than v_0 is in debt. This completes the description of the first step.

Now we are going to explain the second step. Let $D' = D_1$ be the divisor obtained from the first step. While there exists a valid firing strategy f with respect to v_0 such that $(D' - \vec{Q}^T f)(v) \geq 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$, replace D' by $D' - \vec{Q}^T f$. If we show that the procedure terminates, it is obvious that D'

is a v_0 -reduced divisor. Since $f(v_0) = 0$ for any valid firing strategy with respect to v_0 , the vertex v_0 must stop receiving money at some point. At this point none of its neighbors fires, so they must eventually stop receiving money. By iterating this argument we see that, since v_0 is reachable from every vertex, each vertex must stop receiving money at some point. Hence, the above procedure terminates at a v_0 -reduced divisor. \square

Corollary 3.7. *Let D be a divisor satisfying the property (i) in Definition 3.3. Then there exists a sequence of valid firings f_1, \dots, f_k with respect to v_0 such that $D' = D - \vec{Q}^T(\sum_{i=1}^k f_i)$ is v_0 -reduced.*

Lemma 3.8. *For any divisor D , there exist exactly r_0 distinct v_0 -reduced divisors equivalent to D .*

Proof. First, we show that there exist at most r_0 distinct reduced divisors equivalent to D . Suppose not, so by the pigeonhole principle, there exist two distinct reduced divisors, $D' = D - \vec{Q}^T f'$ and $D'' = D - \vec{Q}^T f''$ with $f'(v_0) \equiv f''(v_0) \pmod{r_0}$. Pick $k \in \mathbb{Z}$ so that $(f' - f'' - kR)(v_0) = 0$ and let $f^* = f' - f'' - kR$. By our assumption $D' \neq D''$ and so $\vec{Q}^T(f' - f'') \neq 0$. Hence by Lemma 3.1, either f^* or $-f^*$ satisfies the assumptions of Lemma 3.5. Without loss of generality, suppose f^* satisfies the assumption of Lemma 3.5. But $D' = D'' - \vec{Q}^T f^*$ is a v_0 -reduced divisor, contradicting Definition 3.3(i).

Now, we show that there exist at least r_0 distinct reduced divisors equivalent to D . Lemma 3.6 implies that there exists at least one v_0 -reduced divisor equivalent to D , so if $r_0 = 1$ we are done. Therefore for the rest of the proof we will assume that $r_0 > 1$. Take a v_0 -reduced divisor $D' \sim D$ and observe that $D'' = D' - \vec{Q}^T(\chi_{\{v_0\}})$ satisfies the condition (i) of Definition 3.3. Hence Corollary 3.7 implies that D'' can be reduced without firing v_0 to achieve a new reduced divisor from D' . We can acquire r_0 v_0 -reduced divisors equivalent to D by repeated application of this method. We claim that all of the v_0 -reduced divisors obtained are distinct. Suppose there exist $0 \leq i < j < r_0$ and firing strategies f' and f'' such that $f'(v_0) = i$, $f''(v_0) = j$, and $D^* = D' - \vec{Q}^T f' = D' - \vec{Q}^T f''$ is v_0 -reduced. This implies that $\vec{Q}^T(f'' - f') = \vec{0}$ but $0 < (f'' - f')(v_0) < r_0$, contradicting the statement of Lemma 3.1. \square

Corollary 3.9. *Let \vec{G} be a directed graph and let D be a divisor. There exist r_0 v_0 -reduced divisors $D_i = D - \vec{Q}^T f_i$ where $f_i(v_0) = i$ for all $0 \leq i \leq r_0 - 1$.*

Lemma 3.10. *Let \vec{G} be a directed graph and let D be a divisor. Then*

- (i) *D is equivalent to an effective divisor if and only if there exists a v_0 -reduced divisor $D' \sim D$ such that D' is effective;*
- (ii) *Suppose D is not equivalent to an effective divisor. Then D is an extreme divisor if and only if for any $v \in V(\vec{G})$, there exists a v -reduced divisor $D' \sim D$ such that $D'(v) = -1$.*

Proof. (i): One direction is obvious. So assume D is equivalent to an effective divisor, call it D'' . If D'' is v_0 -reduced then we are done. Otherwise, Corollary 3.7 implies that there exists a valid firing strategy f with respect to v_0 such that $D'' - \vec{Q}^T f$ is v_0 -reduced. Since D'' is effective and f is valid with respect to v_0 , $D'' - \vec{Q}^T f$ is effective.

(ii): First assume that D is an extreme divisor. The assertion of part (i) implies that for all $v \in V(D)$, if $D' \sim D$ is a v -reduced divisor, $D'(v) \leq -1$. Suppose there exists $v \in V(\vec{G})$ such that for all v -reduced divisor $D' \sim D$ we have that $D'(v) < -1$. Then by Remark 3.4, for all v -reduced divisors $D' \sim D$, $D' + \chi_{\{v\}}$ is not effective and it is v -reduced. So by part (i), $D + \chi_{\{v\}}$ is not effective, a contradiction.

For proving the other direction, it is enough to show that for all $v \in V(\vec{G})$, $D + \chi_{\{v\}}$ is equivalent to an effective divisor. So let v be a vertex and let $D' \sim D$ be the v -reduced divisor such that $D'(v) = -1$. Then $D' + \chi_{\{v\}}$ is effective and so $D + \chi_{\{v\}}$ is also. \square

3.1.2 Dhar's Algorithm

Dhar [10], while studying the sand pile model, found a simple algorithm for checking whether a given divisor in an undirected graph G is v_0 -reduced or not. We discuss the directed sandpile model in the next section. Here we generalize his algorithm so that it applies to an arbitrary directed graph \vec{G} . The authors found this generalization independently from Speer [24].

The input of the algorithm is a divisor D satisfying the condition (i) of Definition 3.3. The output of the algorithm is a finite sequence f_i of firing strategies which is decreasing with respect to the \leq relation. The description of the algorithm is as follows.

We construct a sequence of firing strategies f_i 's recursively. Set $f_0 = R$. For $t \geq 0$, if there exists some $v \in V(\vec{G}) \setminus \{v_0\}$ such that

$$(D - \vec{Q}^T f_t)(v) \leq -1, \quad (2)$$

pick one such vertex v and set $f_{t+1} = f_t - \chi_{\{v\}}$. If for all $v \in V(\vec{G}) \setminus \{v_0\}$, $(D - \vec{Q}^T f_t)(v) \geq 0$ and $f_t(v_0) > 0$, set $f_{t+1} = f_t - \chi_{\{v_0\}}$. Otherwise the algorithm terminates and the output of the algorithm is the decreasing sequence of f_i 's.

We call the above algorithm the *generalized Dhar's Algorithm*.

Theorem 3.11. *Let D be a divisor satisfying condition (i) in Definition 3.3. Then*

- (i) *the divisor D is v_0 -reduced if and only if the generalized Dhar's Algorithm terminates at $f_{\vec{1}, R} = \vec{0}$.*
- (ii) *if D is a v_0 -reduced divisor then for each $0 \leq t \leq \vec{1} \cdot R - 1$ such that $f_{t+1} = f_t - \chi_{\{v_0\}}$, $D - \vec{Q}^T f_t$ is a v_0 -reduced divisor.*

Proof. (i): Clearly if D is reduced then the algorithm terminates at $f_{\vec{1}, R} = 0$.

So assume that the algorithm terminates on the divisor D . Take a valid firing f with respect to v_0 and pick t as large as possible such that $f_t \geq f$. The choice of t implies that $f_{t+1} = f_t - \chi_{\{v\}}$ for some vertex $v \in V(\vec{G}) \setminus \{v_0\}$ since $f(v_0) = 0$. Therefore $f_t = f + f'$ where $f' \geq 0$ and $f'(v) = 0$. Hence $(D - \vec{Q}^T f)(v) = (D - \vec{Q}^T f_t - \vec{Q}^T f')(v) \leq (D - \vec{Q}^T f_t)(v) < 0$ so the divisor D satisfies the second condition of Definition 3.3. Hence D is v_0 -reduced.

(ii): For the sake of contradiction, let t be such that $f_{t+1} = f_t - \chi_{\{v_0\}}$ and $D - \vec{Q}^T f_t$ is not a v_0 -reduced divisor. There exists a valid firing strategy f with respect to v_0 such that $((D - \vec{Q}^T f_t) - \vec{Q}^T f)(v) \geq 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. Let $f' = f_t + f$, then we have two cases. Assume there exists $v_i \in V(\vec{G}) \setminus \{v_0\}$ such that $f'(v_i) > r_i$ then $f'' = f' - R$ is a firing strategy which satisfies the conditions of Lemma 3.5, contradicting the fact that for all $v \in V(\vec{G}) \setminus \{v_0\}$, $(D - \vec{Q}^T f')(v) > 0$. Therefore, we can choose s as large as possible such that $f_s \geq f'$. The choice of s implies that there exists $v \in V(\vec{G})$ such that $f_s(v) = f'(v)$ and $f_{s+1} = f_s - \chi_{\{v\}}$. If $v = v_0$, since $t > s$, $f_{s+1} \geq f_t$ but $f_{s+1}(v_0) < f_t$, a contradiction. Hence $v \in V(\vec{G}) \setminus \{v_0\}$ and $(D - \vec{Q}^T f_s)(v) < 0$. But $(D - \vec{Q}^T f')(v) \leq (D - \vec{Q}^T f_s)(v) < 0$ and this contradicts the choice of f and f_t . \square

We conclude this section with the following definition which will appear in each of the subsequent sections.

Definition 3.12. *Let \vec{G} be a directed graph with the Riemann-Roch property. Then \vec{G} has the natural Riemann-Roch property if its canonical divisor K has i th entry $\deg^+(v_i) - 2$ for $0 \leq i \leq n$.*

3.1.3 The Sandpile Model

The sandpile model for a directed graph is a constrained version of the “row” chip-firing game. We define a divisor D to be a v_0 -sandpile configuration if D satisfies the condition (i) from Definition 3.3. The vertex v_0 does not participate in this game and a vertex $v \in V(\vec{G}) \setminus \{v_0\}$ may only fire if it has at least as many chips as its out-degree (so that v does not go in debt), and it never borrows. Moreover, we say that two configurations are the same if they agree at all vertices other than v_0 . This model has been studied

in [12, 16, 24]. The goal of this section is show a connection between the sandpile model and the Riemann-Roch property for the row chip-firing game on a strongly connected directed graph. To do this we will first show a connection between this model and v_0 -reduced divisors. We begin with some necessary definitions.

We now restrict our attention to the sandpile model. We call a v_0 -sandpile configuration v_0 -stable if no vertex $v \in V(\vec{G}) \setminus \{v_0\}$ can fire. We note that while some authors require v_0 to be a global sink (in order to guarantee that a divisor will eventually stabilize), we simply insist that v_0 never fires. We say that a v_0 -sandpile configuration D' stabilizes to D , a v_0 -stable configuration, if D is v_0 -sandpile achievable from D' . To see that any v_0 -sandpile configuration will eventually stabilize to a v_0 -stable configuration, one may follow an argument similar to the one from Lemma 3.6. We note that, as the language suggests, D is unique, i.e., stabilization is independent of the choice of firings, and a simple proof by induction on k , the length of the sequence of firings, gives this fact. A v_0 -stable configuration D is said to be v_0 -reachable from another v_0 -sandpile configuration D' if there exists an effective divisor E such that $D' + E$ stabilizes to D . A v_0 -stable configuration is v_0 -recurrent if it is v_0 -reachable from any other v_0 -sandpile configuration.

Lemma 3.13. *A divisor D is v_0 -recurrent if and only if there exists a divisor D' such that $D'(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ and D' stabilizes to D .*

Proof. We begin with the easier of the two directions. Assume that D is v_0 -recurrent and let D'' be some divisor such that $D''(v) \geq \deg^+(v)$. By definition, D is v_0 -reachable from D'' , therefore there exists some effective divisor E such that $D'' + E = D'$ stabilized to D . This gives the existence of the D' in the statement of the theorem.

Conversely, given some v_0 -sandpile configuration D' such that $D'(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$, which stabilizes to D , we will show that D is v_0 -recurrent. Take some D'' , a v_0 -sandpile configuration. We will show that D is v_0 -reachable from D'' . First let D'' stabilize to the configuration D''' . Now $D''' \leq D'$ so that D is v_0 -reachable from D''' . Let $D' - D''' = E \geq 0$. We claim that $D'' + E$ stabilizes to D . By the observation made above, that stabilization is independent of a choice of firings, it is sufficient to show that there exists a sequence of firings which brings $D'' + E$ to D . Because $D'' + E \geq D''$ we can perform the sequence of firings which brought D'' to D''' . This sequence of firings brings $D'' + E$ to $D''' + E = D'$ and this now stabilizes to D . □

The following definition is for the unconstrained row chip-firing game introduced in the previous section. We say that a divisor D is v_0 -negatively achievable from D' if there exists a sequence of borrowings by individual vertices such that at each step the vertex which borrows has a negative number of chips prior to borrowing.

Lemma 3.14. *A divisor ν is v_0 -reduced if and only if there exists a divisor D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ such that ν is v_0 -negatively achievable from D .*

Proof. We will first show that if ν , a v_0 -sandpile divisor, is v_0 -negatively achievable from D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ then ν is v_0 -reduced. We now introduce some notation, which will be useful for this proof. Let $S : v_{a_1}, \dots, v_{a_k}$ be the sequence of vertices which borrow and let $f_S \leq 0$ be the corresponding firing so that $D - Q^T f_S = \nu$. Let $f_{S,j}$ be the firing strategy defined as $f_{S,j}(v) = |\{i : v_{a_i} = v, i \leq j\}|$ for $1 \leq j \leq k$, with $f_{S,0} = \vec{0}$. Assume that ν is not v_0 -reduced and let $f \neq \vec{0}$ be a natural firing such that $\nu - Q^T f = \nu'$ is a v_0 -sandpile divisor. If $f + f_S \not\leq 0$ then there exists a maximal connected subset A of $V(\vec{G}) \setminus \{v_0\}$ such that $(f + f_S)(v) > 0$ for all $v \in A$, but the set A loses a net positive amount of money via the firing $(f + f_S)$ contradicting the fact that $D - Q^T(f + f_S) = \nu'$ is a v_0 sandpile configuration and $D(v) < 0$ for all $v \in A$. Because $f + f_S \leq 0$ we may take j maximum so that $f_{S,j} \geq f + f_S$ but $f_{S,j+1} \not\geq f + f_S$. This shows that $0 \leq \nu'(v_{a_{j+1}}) = (D - Q^T(f + f_S))(v_{a_{j+1}}) \leq (D - Q^T f_{S,j})(v_{a_{j+1}}) < 0$, a contradiction.

We now show that for any v_0 -reduced divisor ν there exists some D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ such that ν is v_0 -negatively achievable from D . Take ν and greedily fire vertices in $v \in V(\vec{G}) \setminus \{v_0\}$ with an nonnegative number of chips until you obtain D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. To see that this process will eventually terminate adapt the argument give in Lemma 3.6 for why greedy reduction of

a divisor terminates. We claim that D is the desire divisor. If we now, as above, we greedily borrow by vertices in $v \in V(\vec{G}) \setminus \{v_0\}$ which are in debt, we will stop at a v_0 -reduced divisor ν' . To see that this process eventually terminates, again mimic the argument from Lemma 3.6. The fact that ν' is v_0 -reduced was proven above. The divisor ν' is clearly equivalent to ν , and v_0 did not participate in the above process, hence the divisor obtained is equal to ν . \square

The authors, independently from Speer [24], discovered the following theorem.

Theorem 3.15. *A v_0 -sandpile configuration D is v_0 -recurrent if and only if the divisor ν is a v_0 -reduced divisor, where $\nu(v_i) = \deg^+(v_i) - 1 - D(v_i)$ for all $0 \leq i \leq n$.*

Proof. Let K be the divisor such that $K(v_i) = \deg^+(v_i) - 2$. We first note that the map $\phi(D) = K + \vec{1} - D$ is a bijection between divisors D such that $D(v) \geq \deg^+(v)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ and divisors D such that $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. The theorem then follows by observing that ν is v_0 -negatively achievable from D with $D(v) < 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ if and only if $\phi(\nu)$ is v_0 -sandpile achievable from $\phi(D)$ with $(\phi(D))_i \geq \deg^+(v_i)$ for all $v \in V(\vec{G}) \setminus \{v_0\}$. \square

We note that using the notion of equivalence given by the unconstrained row chip-firing game, the previous theorem shows that there are exactly r_0 v_0 -recurrent divisors in each equivalence class. This is different from the case of undirected graphs or directed graphs with v_0 a global sink, where the recurrent state in each equivalence class is unique.

We define a divisor D to be minimally v_0 -recurrent if, ignoring the value of $D(v_0)$, it is minimal with respect to dominance among all v_0 -recurrent divisors. Using this definition we have a new way of describing the natural Riemann-Roch property in terms of the sandpile model for strongly connected directed graphs.

Theorem 3.16. *A directed graph, \vec{G} has the natural Riemann-Roch property if and only if for each minimal v_0 -recurrent divisor D there exists $D' = D + ke_0$, $k \in \mathbb{Z}$, $E_i \in \mathbb{Z}_{\geq 0}$ for $0 \leq i \leq n$ such that $E_i(v_i) = 0$ and $E_i(v_j) > 0$ for $j \neq i$ and $D' \sim E_i$ and each D' is of fixed degree $g - 1 \in \mathbb{N}$.*

Proof. Clearly D is minimally v_0 recurrent if and only if, by Theorem 3.15, we may fix D' as in the statement of the theorem such that $\nu = K - D' + \vec{1}$ is extreme v_0 -reduced. Hence, \vec{G} has the natural Riemann Roch property if and only if $\nu' = D' - \vec{1} \in \text{Ext}(\Sigma(\Lambda))$ and is fixed degree $g - 1$, which occurs precisely when $D' \in \text{Ext}(\Sigma_{\mathbb{R}}(\Lambda))$ and is of fixed degree $g - 1$. By Lemma 2.10, the Theorem follows. \square

3.2 Column Chip-Firing Game, \vec{G} -Parking Functions, and Riemann-Roch Theory

In this section we present a chip-firing game which comes from the columns of the Laplacian matrix.

Definition 3.17. *We call a divisor D a directed \vec{G} -parking function (or simply \vec{G} -parking) with respect to v_0 if the following two conditions hold:*

- (i) *for all $v \in V(\vec{G}) \setminus \{v_0\}$, $D(v) \geq 0$,*
- (ii) *for every set $A \subseteq V(\vec{G}) \setminus \{v_0\}$, there exists some $v \in A$ such that $|\{(v, u) \in E(\vec{G}) : u \notin A\}| \geq D(v)$.*

We introduce the following “column” chip-firing game wherein if a vertex v fires, it loses $\deg^+(v)$ chips and sends a chip along each incoming edge $(u, v) \in E(\vec{G})$ (borrowing is defined as the inverse of firing). Note that the total number of chips is not preserved by firing in contrast to the previous “row” chip-firing game. It is not hard to see that if all vertices in a set A fire once then a vertex $v \in A$ will lose as many chips as it has edges leaving A , i.e., $|\{(v, u) : u \notin A\}|$, while a vertex $u \notin A$ will gain as many chips as it has edges entering to it from A , i.e., $|\{(v, u) : v \in A\}|$. One may view this game as a walk through the lattice spanned by the columns of the Laplacian of \vec{G} and it follows immediately that if D is a divisor then $(D - \vec{Q}\chi_A)(v) = D(v) - |\{(v, u) : u \notin A\}|$ if $v \in A$ and $(D - \vec{Q}\chi_A)(u) = D(u) + |\{(v, u) : v \in A\}|$ if $u \notin A$.

Because \vec{Q} is orthogonal to $\vec{1}$, i.e., $\vec{Q}\vec{1} = \vec{0}$, we have that for any firing strategy f there exists some firing strategy f' such that $\vec{Q}(f - f') = \vec{0}$ and $f' \leq \chi_A$ for some $A \subseteq V(\vec{G}) \setminus \{v_0\}$. It is also worth mentioning that if $R = (r_0, \dots, r_n) \in \mathbb{N}^{n+1}$ is the vector guaranteed by Lemma 3.1 such that $R^T \vec{Q} = \vec{0}^T$, then $\deg_R(\vec{Q}f) = 0$ for all $f \in \mathbb{Z}^{n+1}$, i.e., the total number of chips is preserved in the “column” chip-firing game with respect to $\deg_R(\cdot)$. One may also interpret this fact combinatorially by assigning to each vertex v_i its own “chip currency” worth r_i of a “universal chip currency”. Similar notions of “currencies” and “exchange rates” are employed when discussing chip-firing on arithmetical graphs in Section 4.

A \vec{G} -parking function with respect to v_0 is a divisor D such that $D(v) \geq 0$ for all $v \in V(\vec{G}) \setminus \{v_0\}$ and for each set $A \subset V(\vec{G}) \setminus \{v_0\}$ there exists some vertex $v \in A$ such that $|\{(v, u) : u \notin A\}| > D(v)$. This definition is precisely analogous to the definition of a v_0 -reduced divisors from the “row” chip-firing game. More specifically, if we change \vec{Q}^T to \vec{Q} in definition of v_0 -reduced divisor (Definition 3.3), then we get the definition of \vec{G} -parking function with respect to v_0 (Definition 3.17). Hence, Dhar’s algorithm introduced in [2, 10] applies in verifying whether D is \vec{G} -parking function with respect to v_0 . Note that for undirected graphs the notion of a v_0 -reduced divisor and a G -parking function agree as the Laplacian is symmetric, i.e., the “row” and “column” chip-firing games are identical. It is a well known fact, and has several combinatorial proofs, that the \vec{G} -parking functions are in bijection with set of rooted directed spanning trees [9].

An *Eulerian* directed graph \vec{H} is a directed graph such that $\deg^+(v) = \deg^-(v)$ for each $v \in V(\vec{H})$. The name is derived from the fact that they are exactly those directed graphs which possess a directed Eulerian circuit.

Theorem 3.18. *Let \vec{G} be a strongly connected directed graph with Laplacian \vec{Q} and let \vec{G}' be the Eulerian directed graph with Laplacian $\vec{Q}^T \mathcal{R}$ where $\mathcal{R} = \text{diag}(r_0, \dots, r_n)$ where $\mathcal{R}\vec{1} = \vec{Q}$. The directed graph \vec{G} has the Riemann-Roch property for the column chip-firing game if and only if the directed graph \vec{G}' has the Riemann-Roch property for the row chip-firing game.*

Proof. Let $\Lambda'_{\vec{G}} = \{\vec{Q}f : f \in \mathbb{Z}^{n+1}\}$ be the lattice spanned by the columns of \vec{Q} . It follows by Theorem 2.32 that $\Lambda'_{\vec{G}}$ has the Riemann-Roch property if and only if $\mathcal{R}\Lambda'_{\vec{G}}$ does. This is the lattice spanned by the rows of $\vec{Q}^T \mathcal{R}$ completing the proof. \square

We note that the column chip-firing game for an Eulerian digraph is the same game as the row chip-firing game played on the same directed graph with as of the orientations of all of the arrows reversed. This explains why we are passing to the transpose of the Laplacian in the proof.

Amini and Manjunath [1] have some results related to Eulerian directed graphs (which they call regular digraphs). By the previous theorem, all of these results extend to the column chip-firing game on strongly connected directed graphs. We also remark that for testing whether a divisor is v_0 -reduced, the burning algorithm of Dhar may be applied (burning along incoming edges) and this algorithm can be used to obtain several of the results of Amini and Manjunath related to Eulerian directed graphs.

4 Arithmetical Graphs

Let G be a connected undirected multigraph, choose an ordering $\{v_0, \dots, v_n\}$ of vertices of G , and let A be the corresponding *adjacency matrix* of G . Let $R = (r_0, \dots, r_n)^T \in \mathbb{N}^{n+1}$ be such that $\gcd(r_0, r_1, \dots, r_n) = 1$ and let $\delta_0, \dots, \delta_n \in \mathbb{N}$ be such that $(\mathcal{D} - A)R = \vec{0}$, where $\mathcal{D} = \text{diag}(\delta_0, \dots, \delta_n)$. We say (G, R) is an *arithmetical graph* with Laplacian $Q = \mathcal{D} - A$ and corresponding multiplicity vector R , where for all $0 \leq i \leq n$ the value r_i is the *multiplicity* of the vertex v_i . Note that an undirected graph G can be considered as an arithmetical graph $(G, \vec{1})$.

Consider the following chip-firing game played on the vertices of an arithmetical graph (G, R) . Suppose we have a “universal chip currency” and each vertex v_i has its own “ v_i -chip currency” such that each v_i -chip is worth r_i of the “universal chip currency”. If a vertex v_i *fires*, it loses δ_i of its own v_i -chips and sends $m_{i,j}$ v_j -chips to each v_j adjacent to v_i , where $m_{i,j}$ is the number of edges between v_i and v_j . We define *borrowing* to be the inverse of firing. Let $\Lambda_{(G,R)}$ be the lattice spanned by the columns of Q . It is easy to see that moves

in this chip-firing game correspond to translations of some divisor D by a lattice point $l \in \Lambda_{(G,R)}$. This observation allows us to make use of definitions and theorems from Section 2 when discussing the chip-firing game.

Let (G, R) be an arithmetical graph and $\mathcal{R} = \text{diag}(r_0, \dots, r_n)$. Let \vec{G}_R be the directed graph obtained from (G, R) by replacing each undirected edge (v_i, v_j) with r_j edges directed from v_i to v_j and r_i edges directed from v_j to v_i . The chip-firing game for (G, R) corresponds to the row chip-firing game for \vec{G}_R by converting each vertex's currency to the universal chip currency. If we define \vec{Q}_R be the Laplacian of \vec{G}_R we may observe that $\vec{Q}_R^T = \mathcal{R}Q$. By Theorem 2.32, it follows that the chip-firing game on (G, R) will have the Riemann-Roch property if and only if the row chip-firing game on \vec{G}_R has the Riemann-Roch property. The row chip-firing game on \vec{G}_R is strictly "finer" than the chip-firing game on (G, R) in the sense that a vertex, v_i need not have a multiple of r_i universal chips, although by the previous observation this difference does not effect whether the Riemann-Roch property holds.

In our discussion of the chip-firing game for arithmetical graphs we will borrow several definitions and methods from the row chip-firing game whose interpretation will be clear from the context in which they are used. In particular the definition of a v_0 -reduced divisor and the generalized Dhar's algorithm will be frequently employed.

Theorem 4.1. *Let (G, R) be an arithmetical graph with Laplacian Q and let \vec{G}_R be the associated directed graph. Then \vec{G}_R has the Riemann-Roch property for the column chip-firing game.*

Proof. By Theorem 3.18 it is equivalent to ask the question for the row chip-firing game on the directed graph \vec{H} whose Laplacian is $\mathcal{R}\vec{Q}'$ where \vec{Q}' is the Laplacian for \vec{G}_R . But \vec{Q}' is simply $\vec{Q}\mathcal{R}$ and so \vec{H} has Laplacian $\mathcal{R}\vec{Q}\mathcal{R}$ which as one can easily check is the Laplacian of the undirected graph obtained from G by replacing each edge (v_i, v_j) with $r_i r_j$ edges. By Baker and Norine, this graph has the Riemann-Roch property and this completes the proof. \square

Let $\mathcal{N} = \{D \in \text{Ext}(\Sigma(\Lambda_{(G,R)})) : \deg_R(D) = g_{\max} - 1\}$. For each $0 \leq i \leq n$, let $N(v_i)$ denote the family of vertices which are adjacent to v_i counting their multiplicities. We call $|N(v_i)|$ the *degree* of the vertex v_i and we denote it by $\deg(v_i)$. Recall the definition of g_0 , the number such that $2g_0 - 2 = \sum_{i=0}^n r_i(\delta_i - 2)$. It is not hard to verify and is noted in [18] that g_0 is an integer. It is also easy to see that by firing all of the vertices of the G , we get $\sum_{i=0}^n r_i \delta_i = \sum_{i=0}^n r_i \deg(v_i)$. Therefore $2g_0 - 2 = \sum_{i=0}^n r_i(\deg(v_i) - 2)$.

Theorem 4.2. *Let (G, R) be an arithmetical graph. Then $g_{\max} \leq g_0$.*

Proof. The following proof is an averaging argument employing the generalized Dhar's algorithms and gives a bound twice as good as the naive bound. If one looks closely at the proof, it becomes apparent that arithmetical graphs are precisely those "directed graphs" for which such an averaging argument is effective. Let $D \in \mathcal{N}$. Choose a v_0 -reduced divisor $D' \sim D$ such that $D'(v_0)$ is as large as possible. For proving the theorem, it is enough to show that $\deg_R(D') \leq g_0 - 1$. Apply the generalized Dhar's algorithm to D' . For all $0 \leq i \leq n$ and $1 \leq k \leq r_i$, define $\mathcal{F}_{i,k}$ to be the firing strategy obtained from the generalized Dhar algorithm such that $\mathcal{F}_{i,k}(v_i) = k$ and the successor of $\mathcal{F}_{i,k}$ is the firing strategy $\mathcal{F}_{i,k} - \chi_{\{v_i\}}$. For each $v_i \in V(\vec{G}) \setminus v_0$ we obtain r_i inequalities as follows:
for each k where $1 \leq k \leq r_i$, we have:

$$D'(v_i) \leq k\delta_i - \left(\sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) \right) - 1, \quad (3)$$

which follows from the fact that $(D' - Q\mathcal{F}_{i,k})(v_i) < 0$ by choice of $\mathcal{F}_{i,k}$.
For the vertex v_0 , we know that for all $1 \leq k \leq r_0$,

$$k\delta_0 - \sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \geq 0,$$

by the choice of D' and the second assertion of Lemma 3.11. Because $D' \in \mathcal{N}$, by (ii) of Lemma 3.10 we have that $D'(v_0) < 0$. Hence, for all $1 \leq k \leq r_0$,

$$D'(v_0) \leq k\delta_0 - \left(\sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \right) - 1. \quad (4)$$

Note that $\sum_{i=0}^n \sum_{k=1}^{r_i} D'(v_i) = D' \cdot R = \deg_R(D')$.

Now, taking the sum over all inequalities in (3) and (4), we have:

$$\sum_{i=0}^n \sum_{k=1}^{r_i} D'(v_i) \leq \sum_{i=0}^n r_i((r_i + 1)\delta_i - 2)/2 - \sum_{i=0}^n \sum_{k=1}^{r_i} \sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j). \quad (5)$$

We will now restrict our attention to $\sum_{i=0}^n \sum_{k=1}^{r_i} \sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j)$. By reordering the sums, we have

$$\sum_{i=0}^n \sum_{k=1}^{r_i} \sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) = \sum_{i < j, v_i v_j \in E(G)} \left(\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) \right).$$

We claim that if $v_i v_j \in E(G)$ then $\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_i r_j$. We prove the claim by induction on $r_i + r_j$. If $r_i + r_j = 2$, then the claim holds trivially, since $r_i = r_j = 1$. Now suppose $r_i + r_j = m \geq 3$. Without loss of generality, assume \mathcal{F}_{i,r_i} is generated before \mathcal{F}_{j,r_j} in the run of the generalized Dhar's algorithm on D' . Hence

$$\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_j + \sum_{k=1}^{r_i-1} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = r_j + (r_i - 1)r_j = r_i r_j.$$

The equality $\sum_{k=1}^{r_i-1} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) = (r_i - 1)r_j$ follows from induction hypothesis. This completes the proof of the claim. So

$$\sum_{i < j, v_i v_j \in E(G)} \left(\sum_{k=1}^{r_i} \mathcal{F}_{i,k}(v_j) + \sum_{\ell=1}^{r_j} \mathcal{F}_{j,\ell}(v_i) \right) = \sum_{i < j, v_i v_j \in E(G)} r_i r_j = \frac{1}{2} \left(\sum_{i=0}^n r_i \sum_{v_j \in N(v_i)} r_j \right).$$

Since $QR = 0$, for all $0 \leq i \leq n$, $\sum_{v_j \in N(v_i)} r_j = r_i \delta_i$. Hence

$$\sum_{i=0}^n \sum_{k=1}^{r_i} \sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) = \frac{1}{2} \left(\sum_{i=0}^n r_i^2 \delta_i \right). \quad (6)$$

Now by substituting (6) into inequality (5), we have:

$$\deg_R(D') \leq \sum_{i=0}^n r_i((r_i + 1)\delta_i - 2)/2 - \frac{1}{2} \left(\sum_{i=0}^n r_i^2 \delta_i \right) = \sum_{i=0}^n r_i(\delta_i - 2)/2 = g_0 - 1.$$

□

So the above theorem shows that if, in a configuration of the game identified by $D \in \text{Div}((G, R))$, $\deg_R(D) \geq g_0$, then D has a winning configuration.

Corollary 4.3. $g_{\max} = g_0$ if and only if all inequalities in (3) and (4) obtained in a run of the generalized Dhar's algorithm on a v_0 -reduced divisor $D \in \mathcal{N}$ are tight, i.e. if f_i is the sequence of firing strategies obtained from the run of the generalized Dhar's algorithm on a v_0 -reduced divisor $D \in \mathcal{N}$, for all $0 \leq t \leq \vec{1} \cdot R - 1$, if $f_{t+1} = f_t - \chi_{\{v\}}$ then $(D - Q(f_t))(v) = -1$.

It is clear, and demonstrated below, that if $D \in \mathcal{N}$ and $\deg(D) = g_{\max} - 1$, then for each $v \in V(G)$ and $D' \sim D$ such that D' is v -reduced, we have $D'(v) = -1$. The following theorem shows that the converse is also true.

Theorem 4.4. *Let $D \in \mathcal{N}$. Then $\deg(D) = g_{\max} - 1$ if and only if for each $D' \sim D$ such that D' is a v -reduced divisor, $D'(v) = -1$.*

Proof. Suppose $D \in \mathcal{N}$ with $\deg(D) = g_{\max} - 1$. Take $v \in V(\vec{G})$. By applying (ii) of Lemma 3.10 we may pick $D' \sim D$ to be a v -reduced divisor such that $D'(v) = -1$. Corollary 4.3 implies that all the inequalities are tight, so for all v -reduced divisor $D'' \sim D$, $D''(v) = -1$.

Conversely, assume that $D \in \mathcal{N}$ is v_0 -reduced and suppose that for each $D' \sim D$ which is an extreme v -reduced divisor, $D'(v) = -1$. We wish to show that $\deg(D) = g_{\max} - 1$. Apply the generalized Dhar's algorithm to D , and define $\mathcal{F}_{i,k}$ to be the firing strategy obtained from the generalized Dhar algorithm such that $\mathcal{F}_{i,k}(v_i) = k$ and the successor of $\mathcal{F}_{i,k}$ is the firing strategy $\mathcal{F}_{i,k} - \chi_{\{v_i\}}$.

$$D(v_i) \leq k\delta_i - \left(\sum_{v_j \in N(v_i)} \mathcal{F}_{i,k}(v_j) \right) - 1, \quad (7)$$

which follows from the fact that $(D - Q\mathcal{F}_{i,k})(v_i) < 0$ by choice of $\mathcal{F}_{i,k}$. By the previous corollary, to show that $\deg(D) = g_{\max} - 1$, it is enough to show that each of the inequalities from (7) hold with equality.

For the vertex v_0 , we know that for all $1 \leq k \leq r_0$,

$$k\delta_0 - \sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \geq 0,$$

this follows from the choice of D and the second assertion of Lemma 3.11. Because D is extreme, by (ii) of Lemma 3.10 we have that $D(v_0) < 0$. Hence for all $1 \leq k \leq r_0$,

$$D(v_0) \leq k\delta_0 - \left(\sum_{v_j \in N(v_0)} \mathcal{F}_{0,k}(v_j) \right) - 1. \quad (8)$$

By assumption all of the inequalities for v_0 above hold with equality. So take $v_i \in V(\vec{G}) \setminus v_0$ and $1 \leq k \leq r_i$. For finishing the proof, we will show that $(D - Q(\mathcal{F}_{i,k}))(v_i) = -1$. Let the firing strategy f be such that $D - Qf$ is v_i -reduced and $f(v_i) = k$, where the existence of f is guaranteed by Corollary 3.9. Assume $f' \approx f$ is a natural firing strategy. Let f_t 's be the sequence of firing strategies obtained from a run of the generalized Dhar's algorithm on D . Take j as large as possible such that $f_j \geq f'$. Let $v \in V(\vec{G})$ be such that $f_{j+1} = f_j - \chi_{\{v\}}$. Let the firing strategy f'' be such that $f' = f_j - f''$ where $f'' \geq \vec{0}$ and $f''(v) = 0$. We claim that $v = v_i$. If $v \notin \{v_0, v_i\}$ then $(D - Qf')(v) = (D - Q(f_j - f''))(v) \leq (D - Q(f_j))(v) < 0$, contradicting the fact that $D - Qf'$ is a v_i -reduced. If $v = v_0$, then $(D - Qf')(v_0) = (D - Q(f_j - f''))(v_0) \leq (D - Q(f_j))(v_0) = -1$ since $D - Qf_j$ is a v_0 -reduced divisor by the second part of Theorem 3.11. But this again contradicts the fact that $D - Qf'$ is a v_i -reduced divisor. Hence $v = v_i$ and this finishes the proof of the claim. Therefore $f_j = \mathcal{F}_{i,k}$ and we have:

$$-1 = (D - Qf')(v_i) = (D - Q(f_j - f''))(v_i) = (D - Q(\mathcal{F}_{i,k} - f''))(v_i) \leq (D - Q(\mathcal{F}_{i,k}))(v_i) \leq -1.$$

Hence $(D - Q(\mathcal{F}_{i,k}))(v_i) = -1$ as desired. \square

We note that a more general version of the previous theorem can be stated for strongly connected directed graphs and might have been included in the section on Dhar's algorithm, but because we do not have statement like Corollary 4.3 for all strongly connected directed graphs, the statement of this more general theorem would have been awkwardly phrased.

Theorem 4.5. Let $K = (\delta_0 - 2, \dots, \delta_n - 2)$ be a vector in \mathbb{Z}^{n+1} . If $g_{\max} = g_0$ then $D \in \mathcal{N}$ if and only if $K - D \in \mathcal{N}$.

Proof. Without loss of generality, we may assume D is a v_0 -reduced divisor. Apply the generalized Dhar's algorithm on D and let f_i be the output sequence. Let $\mathcal{F}_{i,k}$ be the firing strategies defined in the proof of Theorem 4.2.

Define the divisor D' such that for all $0 \leq i \leq n$,

$$D'(v_i) = k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) \right) - 1.$$

We claim that D' is well-defined. For proving the claim, it is enough to show that for all $0 \leq i \leq n$, the value of $D'(v_i)$ does not depend upon k . We will show $D' = K - D$.

Since $g_{\max} = g_0$, Corollary 4.3 implies that for all $0 \leq i \leq n$, $\sum_{v_j \in N(v_i)} \mathcal{F}_{i,r_i+1-k}(v_j) = (r_i + 1 - k)\delta_i - D(v_i) - 1$. For all $0 \leq i \leq n$, we have:

$$\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) = \left(\sum_{v_j \in N(v_i)} r_j \right) - ((r_i + 1 - k)\delta_i - D(v_i) - 1) = -\delta_i + k\delta_i + D(v_i) + 1.$$

Therefore,

$$D'(v_i) = k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) \right) - 1 = k\delta_i - (-\delta_i + k\delta_i + D(v_i) + 1) - 1 = \delta_i - 2 - D(v_i).$$

Since $\deg_R(K - D) = g_0 - 1$, for finishing the proof we only need to show that $K - D$ is not equivalent to an effective divisor.

Assume to the contrary that D' is equivalent to some effective divisor E and let f be such that $D' - Qf = E$. Let $f' \approx f$ be a natural firing strategy guaranteed by Lemma 3.2. Define a “reverse sequence” of firing strategies $f'_i = R - f_{\vec{1}, R-i}$ for all $0 \leq i \leq \vec{1} \cdot R$. Take t as large as possible such that $f'_t \geq f'$. So there exists $v_i \in V(\vec{G})$ such that $f'(v_i) = f'_t(v_i)$. By the definition of the reverse sequence, there exists $1 \leq k \leq r_i$ such that $f'_t = R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}}$. Therefore,

$$\begin{aligned} E(v_i) &\leq (D' - Qf'_t)(v_i) \\ &= k\delta_i - \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j) \right) - 1 - (r_i - (r_i + 1 - k) - 1)\delta_i + \left(\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}})(v_j) \right) \\ &= k\delta_i - (r_i - (r_i + 1 - k) - 1) - 1 = -1. \end{aligned}$$

Note that $\sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k} + \chi_{\{v_i\}})(v_j) = \sum_{v_j \in N(v_i)} (R - \mathcal{F}_{i,r_i+1-k})(v_j)$. This contradicts the choice of E . Hence $D' = K - D$ is not equivalent to an effective divisor. \square

We should mention that Theorem 4.2 and Theorem 4.5 are due to Lorenzini [17]. His approach in proving these theorems is purely algebraic. As mentioned in [17], he was interested in combinatorial proof of these facts which could be the one presented in this paper.

Theorem 4.6. Let (G, R) be an arithmetical graph. If $g_0 = g_{\min} = g_{\max}$, then (G, R) has the Riemann-Roch property. Moreover, the corresponding directed graph has the natural Riemann-Roch property.

Proof. The first part of the theorem follows as an immediate consequence of Theorem 2.26 and Theorem 4.5. The second part of the theorem follows by Corollary 2.31, which in this context says that if $g_0 = g_{\min} = g_{\max}$, then the canonical divisor for the corresponding digraph \vec{G}_R has i th entry $\deg^+(v_i) - 2$, i.e., \vec{G}_R satisfies Definition 3.12 for the row chip-firing game. Moreover, we note that $(\delta_0 - 2, \dots, \delta_n - 2) \sim (\deg(v_0) - 2, \dots, \deg(v_n) - 2)$ as is easily observed by computing $Q\vec{1}$. \square

Corollary 4.7. *Let (G, R) be an arithmetical graph. If $\Lambda_{(G, R)}$ has a unique class of extreme divisors, i.e. $\text{Ext}(\Sigma(\Lambda_{(G, R)})) = \{\nu + \ell : \ell \in \Lambda_{(G, R)}\}$, then $\Lambda_{(G, R)}$ has the Riemann-Roch property.*

4.1 Arithmetical Graphs with the Riemann-Roch Property

Theorem 4.8. *Let (G, R) be an arithmetical graph. If $g_0 \leq 1$ then (G, R) has the Riemann-Roch property.*

Proof. Let v_0 be a vertex such that $r_0 \leq r_i$ for all $1 \leq i \leq n$. Let D be an extreme v_0 -reduced divisor with $D(v_0) = -1$. By Theorem 4.2 $g_{\max} \leq g_0$, so $\deg(D) \leq g_{\max} - 1 \leq 0$. Now we have two cases:

- (i) $D(v_i) = 0$ for all $1 \leq i \leq n$, part (ii) of Lemma 3.10 and the choice of r_0 implies that D is the unique extreme v_0 -reduced divisor, and the assertion of the lemma holds by Corollary 4.7. Note that in this case $g_{\max} \neq g_0$ unless $g_0 = 0$ and $r_0 = 1$.
- (ii) There exists $1 \leq i \leq n$ such that $D(v_i) > 0$. Since $\deg(D) \leq 0$, $r_i = r_0$ and v_i is the only vertex with $D(v_i) > 0$. This implies that the divisor D' with $D'(v_0) = -1$ and $D'(v_j) = 0$ for all $1 \leq j \leq n$ is not an extreme divisor. Hence, $g_0 = g_{\min} = g_{\max} = 1$, and assertion of the lemma follows by Theorem 4.6. \square

Using the definition of g_0 the following is immediate consequence of the Theorem 4.8.

Corollary 4.9. *Let (G, R) be an arithmetical graph with all δ_i 's equal to two or all $\deg(v_i)$'s equal to two. Then (G, R) has the Riemann-Roch property.*

The former arithmetical graphs are those coming from the connection between Lie algebras or elliptical curves which have been classified [7] and the latter arithmetical graphs where the underlying graph is a cycle. The following two examples show that both cases described in the proof of Theorem 4.8 occur.

Example 4.10. *Let (G, R) be an arithmetical graph where G is the even cycle v_0, \dots, v_{2n-1} for $n \geq 2$, and for all $0 \leq i \leq n-1$, the multiplicities of the vertices v_{2i} and v_{2i+1} are 1 and 2, respectively. Then $g_{\min} = g_{\max} = g_0 = 1$, and in particular (G, R) has the Riemann-Roch property.*

Proof. We claim that the set of extreme v_0 -reduced divisors for (G, R) are the set of divisors $D_i = \chi_{\{v_{2i}\}} - \chi_{\{v_0\}}$ for all $1 \leq i \leq n-1$. Assume $1 \leq i \leq n-1$, and the vector f is a valid firing strategy with respect to v_0 such that $D_i - Qf \geq \vec{0}$. Observe that if $f(v_{2i}) = 1$, then in order to $(D_i - Qf)(v_{2i}) \geq 0$ we must have $f(v_{2i-1}) + f(v_{2i+1}) \geq 3$. By symmetry, assume that $f(v_{2i-1}) \geq 2$. Since $(D_i - Qf)(v_{2i-1}) \geq 0$, we have $f(v_{2i-2}) = 1$. By repeating the argument, we conclude that $f(v_0) = 1$, a contradiction. This shows that D_i is v_0 -reduced and since $r_0 = 1$, (i) of Lemma 3.10 implies that D_i is not equivalent to an effective divisor. For proving the fact that D_i is an extreme divisor, it is enough to show that $D_i + \chi_{\{v_j\}}$ is equivalent to an effective divisor, for all $0 \leq j \leq 2n-1$.

It is easy to see that $g_0 = 1$. If $0 \leq j \leq 2n-1$ is odd, then the divisor $D_i + \chi_{\{v_j\}}$ has degree $2 > g_0$, thus Theorem 4.2 implies that $D_i + \chi_{\{v_j\}}$ is effective. We claim that for all $0 \leq j \leq i \leq n-1$, the divisor $D_i + \chi_{\{v_{2j}\}}$ is equivalent effective. We prove the claim by induction on j . If $j = 0$, then the assertion of the claim trivially holds. So, assume $j > 0$ and let $f = \chi_{\{v_{2j-1}, \dots, v_{2i+1}\}}$. A simple computation gives that $D_i + \chi_{\{v_{2j}\}} - Qf = D_{i+1} + \chi_{\{v_{2j-2}\}}$. The induction hypothesis implies that $D_{i+1} + \chi_{\{v_{2j-2}\}}$ is equivalent an effective divisor, so is $D_{i+1} + \chi_{\{v_{2j}\}}$. This shows that D_i 's are extreme v_0 -reduced divisors.

Now assume that D is an extreme v_0 -reduced divisor. Part (ii) of Lemma 3.10 implies that $D(v_0) = -1$. If $D(v_{2i+1}) = 1$ for some $0 \leq i \leq n-1$, then D is not a v_0 -reduced divisor. The above argument shows that if $D(v_{2i}) = 2$ or $D(v_{2i}) = D(v_{2j}) = 1$ for some $0 \leq i \neq j \leq n-1$, the divisor D is equivalent to an effective divisor. Obviously $D \neq -\chi_{\{v_0\}}$, and this completes the proof of the claim.

Since each extreme v_0 -reduced divisor D_i , $1 \leq i \leq n-1$ has degree zero, $g_{\min} = g_{\max} = g_0$. Theorem 4.6 implies that (G, R) has the Riemann-Roch property. \square

Example 4.11. *Let (G, R) be an arithmetical graph where G is a cycle v_1, \dots, v_n for $n \geq 3$ and the multiplicity of vertex v_i is i for all $1 \leq i \leq n$. Then (G, R) has Riemann-Roch property.*

Proof. It is easy to see that $g_0 = 1$. Now assume D is an extreme v_1 -reduced divisor. The part (ii) of Lemma 3.10 implies that $D(v_1) = -1$. If there exists $2 \leq i \leq n$ such that $D(v_i) \geq 1$, then degree of D is at least one. Thus, Theorem 4.2 implies that D is equivalent to an effective divisor. This shows that $D = -\chi_{\{v_1\}}$ is the unique extreme v_1 -reduced divisor and the assertion of the lemma follows Corollary 4.7. \square

The following example introduced in [17] has the Riemann-Roch property.

Example 4.12. Let (G, R) be an arithmetical graph where G is a graph with vertex set $\{v_0, v_1\}$ such that v_0 is connected to v_1 with $r_0 r_1$ edges where r_0 and r_1 are the multiplicity of the vertex v_0 and v_1 , respectively. Then (G, R) has the Riemann-Roch property.

Proof. The proof follows from Corollary 4.7, since there exists a unique extreme v_0 -reduced divisor, $D = -\chi_{\{v_0\}} + (r_0^2 - 1)\chi_{\{v_1\}}$. Hence $g_{\min} = g_{\max} = g_0$. \square

Given any two integers $r_0 > r_1$ we can recursively construct a decreasing sequence r_i 's where $r_{i+1} = \delta_i r_i - r_{i-1}$, $r_{i+1} < r_i$ and $\delta_i \in \mathbb{N}$ for all $i \geq 1$. We call such a sequence the *Euclidean sequence generated by r_0 and r_1* . Note that the Euclidean sequence generated by r_0 and r_1 is finite and it comes from a simple variation of Euclid's algorithm.

Let (G, R) be an arithmetical graph. We define a *Euclidean chain leaving v_0 generated by r_0 and r_1* to be an induced path $C = v_0, v_1, \dots, v_n$ of length $n + 1 \geq 2$ in G such that $\deg_G(v_n) = 1$ where the corresponding sequence of multiplicities, r_0, r_1, \dots, r_n is the Euclidean sequence generated by r_0 and r_1 . Note that $r_n = \gcd(r_i, r_{i+1})$ for all $0 \leq i \leq n - 1$. If v_0, r_0 and r_1 are clear from the context, we may simply refer to the path as a *Euclidean chain*.

Lorenzini [18] uses a slight variation of the Euclidean chain for building arithmetical graphs. We also use Euclidean chain to construct a arithmetical graph with the Riemann-Roch property.

A *Euclidean star generated by r_0 and r_1* is an arithmetical graph (G, R) with the *center* vertex v_0 with multiplicity r_0 and r_0 identical Euclidean chains leaving v_0 generated by r_0 and r_1 . We call the vertex v_0 the *center vertex*. When r_0 and r_1 are clear from the context, we will simply say *Euclidean star*.

We will show that every Euclidean star generated by r_0 and r_1 with $\gcd(r_0, r_1) = 1$, has the Riemann-Roch property.

Definition 4.13. Let $r_0 > r_1$ be two positive integers with $\gcd(r_0, r_1) = 1$. Assume r_0, r_1, \dots, r_m is the Euclidean sequence generated by r_0 and r_1 . Given a nonnegative integer x , we say x has a *good representation with respect to r_0 and r_1* if there exist $0 \leq t_i \leq \delta_i - 1$, for all $1 \leq i \leq m$ such that $x = \sum_{i=1}^m t_i r_i$, and there exist no $1 \leq i < j \leq m$ such that $t_i = \delta_i - 1$, $t_j = \delta_j - 1$ and for all $i < k < j$, $t_k = \delta_k - 2$.

Lemma 4.14. Let r_0 and r_1 be positive integers with $\gcd(r_0, r_1) = 1$. Given a nonnegative integer x , x has a good representation with respect to r_0 and r_1 if and only if $0 \leq x \leq r_0 - 1$. Moreover, if $0 \leq x \leq r_0 - 1$ such a representation is unique.

Proof. Assume r_0, r_1, \dots, r_m is the Euclidean sequence generated by r_0 and r_1 . We prove by induction on m . If $m = 1$, the assertion of the lemma is obvious. Now assume $m \geq 2$ and x is an arbitrary nonnegative integer. It is easy to see that $t_1 \leq \lfloor \frac{x}{r_1} \rfloor$. If $t_1 < \lfloor \frac{x}{r_1} \rfloor$, then $x - t_1 r_1 \geq r_1$, so by the induction hypothesis $x - t_1 r_1$ does not have a good representation with respect to r_1 and r_2 because $\gcd(r_1, r_2) = 1$ and the Euclidean sequence obtained from r_1 and r_2 is r_1, r_2, \dots, r_m .

Hence, we may assume $t_1 = \lfloor \frac{x}{r_1} \rfloor$, so by induction hypothesis $x - t_1 r_1$ has a good representation with respect to r_1 and r_2 . If $t_1 \leq \delta_1 - 2$, then the good representation of $x - t_1 r_1$ with respect to r_1 and r_2 extends to a good representation of x with respect to r_0 and r_1 .

If $t_1 = \delta_1 - 1$, then $x - (\delta_1 - 1)r_1 = x - r_0 - r_2 + r_1 < r_1 - r_2$, therefore $x - t_1 r_1 + r_2 = \sum_{i=2}^m t_i r_i$ is a unique good representation with respect to r_1 and r_2 . We claim $t_2 \geq 1$. If $t_2 = 0$ then $x - t_1 r_1 + r_2$ has a good representation with respect to r_2 and r_3 , therefore by induction $x - t_1 r_1 + r_2 < r_2$, so $x - t_1 r_1 < 0$, a contradiction. Therefore $(t_2 - 1)r_2 + \sum_{i=3}^m t_i r_i$ is the unique good representation of $x - t_1 r_1$ with respect to r_1 and r_2 . We claim that $t_1 r_1 + (t_2 - 1)r_2 + \sum_{i=3}^m t_i r_i$ is the unique good representation of x with respect to

r_0 and r_1 . Uniqueness has been established so it remains to show that the representation is good. Assume the representation is not good. It follows that there exists $i \geq 3$ such that $t_i = \delta_i - 1$ and for all $2 < k < i$, $t_k = \delta_k - 2$, and $t_2 - 1 = \delta_2 - 2$. Therefore, $t_2 = \delta_2 - 1$, which implies $\sum_{i=2}^m t_i r_i$ is not a good representation of $x - t_1 r_1 + r_2$ with respect to r_0 and r_1 , a contradiction.

Suppose there exists an integer $x \geq r_0$ such that x has a good representation with respect to r_0 and r_1 , $x = \sum_{i=1}^m t_i r_i$. If $t_1 \leq \delta_1 - 2$ then $x - t_1 r_1 \geq x - (r_0 + r_2) + 2r_1 \geq r_1$. So by induction hypothesis $x - t_1 r_1$ does not have a good representation respect to r_1 and r_2 , a contradiction. Hence $t_1 = \delta_1 - 1$ and $x - t_1 r_1 < r_1$. This implies that $x - t_1 r_1 \geq x - (r_0 + r_2) + r_1 \geq r_1 - r_2$. Let $x - t_1 r_1 = \sum_{i=2}^m t_i r_i$ be the good representation of $x - t_1 r_1$ with respect to r_1 and r_2 . By induction hypothesis $x - t_1 r_1 + r_2 \geq r_1$ does not have a good representation with respect to r_1 and r_2 . Either there exists $3 \leq j \leq m$ such that $t_j = \delta_j - 1$, $t_2 + 1 = \delta_2 - 1$ and $t_i = \delta_i - 2$ for all $2 < i < j$, or $t_2 + 1 = \delta_2$, both of which contradict the fact that $\sum_{i=1}^m t_i r_i$ is a good representation of x with respect to r_0 and r_1 because $t_1 = \delta_1 - 1$. \square

Lemma 4.15. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with center vertex v_0 . Then the set of all v_0 -reduced divisors are the set of divisors such that for any Euclidean chain $C = v_0, v_1, \dots, v_m$ leaving v_0 , $x = \sum_{i=1}^m D(v_i) r_i$ is a good representation with respect to r_0 and r_1 .*

Proof. Let D be a v_0 -reduced divisor and $C = v_0, v_1, \dots, v_m$ be a Euclidean chain leaving v_0 . It is clear that if $x = \sum_{i=1}^m D(v_i) r_i$ is not a good representation with respect to r_0 and r_1 then D is not a v_0 -reduced divisor.

Conversely, let D be a divisor such that for every Euclidean chain $C = v_0, v_1, \dots, v_m$ leaving v_0 , $x = \sum_{i=1}^m D(v_i) r_i$ is a good representation with respect to r_0 and r_1 , but D is not a v_0 -reduced divisor. Let $f \geq \vec{0}$ be a firing strategy such that $f(v_0) = 0$ and $D' = D - Qf$ is a v_0 -reduced divisor. Note that the existence of f is guaranteed by Corollary 3.7. Let $C = v_0, v_1, \dots, v_m$ be a Euclidean chain leaving v_0 . Without loss of generality we may assume $f' \neq \vec{0}$ where f' is the projection of f into the first $m + 1$ coordinates. If $f'(v_1) > 0$ then $\sum_{i=1}^m D'(v_i) r_i < 0$, therefore there exists $1 \leq i \leq m$ such that $D'(v_i) < 0$, a contradiction. Hence, $\sum_{i=1}^m D'(v_i) r_i = \sum_{i=1}^m D(v_i) r_i$. Since $f' \neq \vec{0}$, by Lemma 3.1 and the uniqueness of the representation of $\sum_{i=1}^m D(v_i) r_i$ implied by Lemma 4.14, $\sum_{i=1}^m D'(v_i) r_i$ is not a good representation. Therefore D' is not v_0 -reduced, a contradiction. \square

Definition 4.16. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with the center vertex v_0 . We say a divisor S is a staircase divisor if there exists a labeling C_0, \dots, C_{r_0-1} of the Euclidean chains leaving v_0 where $P_i = v_0, v_{i,1}, \dots, v_{i,m}$ is the induced path of C_i such that $\sum_{j=1}^m S(v_{i,j}) r_j$ is the good representation of i , for all $0 \leq i \leq r_0 - 1$, and $S(v_0) = -1$.*

Lemma 4.17. *Let (G, R) be a Euclidean star generated by r_0 and r_1 with the center vertex v_0 . A divisor D is an extreme v_0 -reduced divisor if and only if D is a staircase divisor.*

Proof. Let S be a staircase divisor and C_0, \dots, C_{r_0-1} be a labeling of the Euclidean chains leaving v_0 where $v_0, v_{i,1}, \dots, v_{i,m}$ are the vertices of C_i . We claim that S is not equivalent to an effective divisor. For proving the claim, it is enough to show that all v_0 -reduced divisors equivalent to S are staircase divisors. Let $1 \leq k \leq r_0$ and f_k be the firing strategy guaranteed by Corollary 3.9, such that $f_k(v_0) = k$ and $S_k = S - Qf_k$ is a v_0 -reduced divisor. Note that since S is a v_0 -reduced divisor, by Lemma 4.15, the divisor S is v_0 -reduced. So, as an application of part (ii) of Theorem 3.11, we may assume $f_k \geq \vec{0}$. It is clear from the proof of Lemma 4.15, $\sum_{j=1}^m S_k(v_{i,j}) r_j$ is a good representation of $i + k r_1 \bmod r_0$ for all $0 \leq i \leq r_0 - 1$. Note that S_k is a staircase divisor and $s_k(v_0) = -1$. So (i) of Lemma 3.10 implies that S_k is not equivalent to an effective divisor.

Now, we prove that for any v_0 -reduced divisor D not equivalent to an effective, there exists a staircase divisor S such that $D' \sim D$ such that $D' \leq S$. Let C_0, \dots, C_{r_0-1} be a labeling of the Euclidean chains leaving v_0 where $v_0, v_{i,1}, \dots, v_{i,m}$ are the vertices of C_i such that $\sum_{j=1}^m D(v_{i,j}) r_j \leq \sum_{j=1}^m D(v_{i+1,j}) r_j$ for all $0 \leq i \leq r_0 - 2$. Let S be the staircase divisor defined by the same labeling of the Euclidean chains leaving v_0 . If for all $0 \leq i \leq r_0 - 1$, $\sum_{j=1}^m D(v_{i,j}) r_j \leq i$ then $D \leq S$, so we may assume that there exists $0 \leq i \leq r_0 - 1$ such that $\sum_{j=1}^m D(v_{i,j}) r_j > i$. Let k be such that $k r_1 \equiv r_0 - i - 1 \pmod{r_0}$. By Corollary 3.9 there exist

firing strategies f_D and f_S such that $f_D(v_0) = f_S(v_0) = k$ and the divisors $D_k = D - Qf_D$ and $S_k = S - Qf_S$ are v_0 -reduced. We claim that D_k is effective, in particular $D_k(v_0) = 0$. We have $f_D(v_{\ell,1}) = f_S(v_{\ell,1}) = \lfloor \frac{kr_1}{r_0} \rfloor$ for all $0 \leq \ell \leq i-1$ and $f_D(v_{\ell,1}) = f_S(v_{\ell,1}) = \lceil \frac{kr_1}{r_0} \rceil$ for all $i+1 \leq \ell \leq r_0-1$, but $f_D(v_{i,1}) = \lceil \frac{kr_1}{r_0} \rceil$ while $f_S(v_{i,1}) = \lfloor \frac{kr_1}{r_0} \rfloor$. This proves the claim and completes the proof of the lemma. \square

Theorem 4.18. *Let (G, R) be a Euclidean star then (G, R) has the Riemann-Roch property.*

Proof. By Lemma 4.17, we know that the set of staircase divisors is the set of extreme v_0 -reduced divisors, hence

$$g_{\min} - 1 = g_{\max} - 1 = \left(\sum_{i=0}^{r_0-1} i \right) - r_0 = r_0(r_0 - 3)/2.$$

Let $V(\vec{G}) = \{v_0, \dots, v_n\}$. Using the formula

$$g_0 - 1 = \sum_{i=0}^n r_i(\deg(v_i) - 2)/2 = r_0(r_0 - 3)/2 = \binom{r_0 - 1}{2} - 1.$$

Now the assertion of the theorem follows from Theorem 4.6. \square

4.2 Arithmetical Graphs without the Riemann-Roch Property

It follows from Theorem 2.26 that an arithmetical graph (G, R) fails to have the Riemann-Roch property if (G, R) is not uniform or is not reflection invariant. The following examples show that all of these three possibilities can happen.

Example 4.19. *Let (G, R) be an arithmetical graph, where G is the graph obtained by adding two edges connecting v_0 to v_3 to the 6-cycle v_0, \dots, v_5 , and the multiplicity of the vertex v_i is 1 if $i \in \{0, 2, 4\}$ and is 2 otherwise. Then (G, R) is neither uniform nor reflection invariant.*

Proof. Let $\nu_1 = -\chi_{\{v_0\}} + \chi_{\{v_2, v_3, v_4\}}$, $\nu_2 = -\chi_{\{v_0\}} + \chi_{\{v_2\}} + 2\chi_{\{v_4\}}$ and $\nu_3 = -\chi_{\{v_0\}} + 2\chi_{\{v_2\}} + \chi_{\{v_4\}}$. We claim that $\mathcal{E} = \{\nu_1, \nu_2, \nu_3\}$ is the set of extreme v_0 -reduced divisors of (G, R) . Note that $\deg_R(\nu_1) = 3$ and $\deg_R(\nu_2) = \deg_R(\nu_3) = 2$. For proving the claim we start by showing that ν_1 is v_0 -reduced. Let f be a valid firing strategy with respect to v_0 such that $(D_1 - Qf)(v_i) \geq 0$, for all $1 \leq i \leq 5$. If $f(v_2) = 1$, since $(D_1 - Qf)(v_2) \geq 0$, we have $f(v_1) + f(v_3) \geq 3$. If $f(v_1) = 2$, since $(D_1 - Qf)(v_1) \geq 0$ we must have $f(v_0) \geq 1$, a contradiction. So $f(v_3) = 2$ and this implies that in order to have $(D_1 - Qf)(v_3) \geq 0$ we must have $f(v_4) = 3$, a contradiction. This shows that $f(v_1) = 0$, and by symmetry $f(v_5) = f(v_4) = 0$, which shows that $f(v_3) = 0$. This shows that $f = \vec{0}$, which contradicts the fact that f is valid strategy with respect to v_0 . Hence, ν_1 is v_0 -reduced, as desired. By applying a similar argument, we can see that ν_2 and ν_3 are v_0 -reduced divisors. Note that since $r_0 = 1$, by Lemma 3.10(i), the v_0 -reduced divisors ν_1, ν_2, ν_3 are not effective and they are pairwise inequivalent.

It is easy to compute that $\deg_R(\nu_1) = 3 = g_0 - 1$, so Theorem 4.2 implies that ν_1 is extreme. Hence, by symmetry, we only need to prove that ν_2 is extreme. For proving this fact it is enough to show that $D = \nu_2 + \chi_{\{v_i\}}$ is equivalent to an effective divisor for all $0 \leq i \leq 5$. If $i \notin \{0, 2, 4\}$, then degree of D is $4 = g_0$, so Theorem 4.2 implies that D is equivalent to an effective divisor. If $i = 0$, then D is trivially effective. If $i = 2$, then we have a firing strategy $f = \vec{1} - \chi_{\{v_0\}}$ such that $D - Qf = 3\chi_{\{v_0\}} \geq \vec{0}$. Also if $i = 4$, then we have $f = \chi_{\{v_4, v_5\}}$ such that $D - Qf = \chi_{\{v_2, v_3\}} \geq \vec{0}$. This completes the proof of the fact that ν_1, ν_2, ν_3 are extreme v_0 -reduced divisors.

Suppose ν is an extreme v_0 -reduced divisor. It is easy to see that $\nu(v_2) \leq 2$ (by symmetry $\nu(v_4) \leq 2$), since otherwise $\nu - Qf \geq 0$, where $f = \chi_{\{v_1, v_2\}}$. Note that $\nu(v_1) = \nu(v_5) = 0$ and $\nu(v_3) \leq 1$. It follows that \mathcal{E} is the set of v_0 -reduced divisors and this completes the proof of the claim. This demonstrates that (G, R) is not uniform.

Now, we are going to show that (G, R) is not reflection invariant. Let Λ be the lattice spanned by Laplacian of (G, R) . By applying Lemma 3.6 and (ii) of Lemma 3.10, we conclude that $\text{Ext}(\Sigma(\Lambda)) =$

$\{\nu + \ell : \ell \in \Lambda, \nu \in \mathcal{E}\}$. Corollary 2.14 implies $\text{Crit}(\Lambda) = \mathcal{P} + \Lambda$, where $\mathcal{P} = \{\pi(\nu + \vec{1}) : \nu \in \mathcal{E}\}$. Let $p_i = \pi(\nu_i + \vec{1}) = (\nu_i + \vec{1}) - \left(\frac{(\nu_i + \vec{1}) \cdot R}{R \cdot R}\right) R$. An easy computation shows that $p_1 = \frac{1}{5}(-4, -3, 6, 2, 6, -3)$, $p_2 = \frac{1}{15}(-11, -7, 19, -7, 34, -7)$ and $p_3 = \frac{1}{15}(-11, -7, 34, -7, 19, -7)$. For seeking a contradiction, assume there exists $v \in \mathbb{R}^6$ such that $-\text{Crit}(\Lambda) = \text{Crit}(\Lambda) + v$. Either there exist $\ell, \ell', \ell'' \in \Lambda$ such that $-p_1 = p_1 + \ell + v$, $-p_2 = p_2 + \ell' + v$ and $-p_3 = p_3 + \ell'' + v$, in this case $2(p_i - p_j) \in \Lambda$ for all $1 \leq i \neq j \leq 3$. Or, there exist $\ell, \ell' \in \Lambda$ and $\{i, j, k\} = \{1, 2, 3\}$ such that $-p_i = p_j + \ell + v$, and $-p_k = p_k + \ell' + v$, in this case $-p_j = p_i + \ell + v$ and we must have $-2p_k + p_i + p_j \in \Lambda$. Note that $\Lambda \subseteq \mathbb{Z}^6$, so an easy computation shows that none of the above cases happen. This proves that (G, R) is not reflection invariant. \square

Example 4.20. Let (G, R) be an arithmetical graph, where G is a graph obtained from K_4 where $V(K_4) = \{v_0, v_1, v_2, v_3\}$, by subdividing the edge v_2v_3 twice. The multiplicity of the vertices v_0 and v_1 are 2 and 4 respectively, and the multiplicity of the other vertices are 3. Then (G, R) is uniform but not reflection invariant.

Proof. Let $P = v_2v_4v_5v_3$ be the induced path connecting v_2 to v_3 , i.e., the path obtained by subdividing the edge v_2v_3 in the graph K_4 .

Let $\nu_1 = -\chi_{\{v_0\}} + \chi_{\{v_2, v_4\}}$, $\nu_2 = -\chi_{\{v_0\}} + 2\chi_{\{v_2\}}$ and $\nu_3 = -\chi_{\{v_0\}} + 2\chi_{\{v_3\}}$. We claim that $\mathcal{E} = \{\nu_1, \nu_2, \nu_3\}$ is the set of extreme v_0 -reduced divisors of (G, R) . By running the Generalized Dhar's Algorithm on each ν_i , $1 \leq i \leq 3$, it is not hard to see that $\nu_1 \sim -\chi_{\{v_0\}} + \chi_{\{v_3, v_5\}}$, $\nu_2 \sim -\chi_{\{v_0\}} + \chi_{\{v_3, v_4\}}$ and $\nu_3 \sim -\chi_{\{v_0\}} + \chi_{\{v_2, v_5\}}$.

We will leave the details of the fact that ν_i , $1 \leq i \leq 3$ is v_0 -reduced to the reader. (It follows from Lemma 3.11, or case analysis similar to that one used in the proof of the Example 4.19.) It is easy to compute that $g_0 = 7$, and for all $\nu \in \mathcal{E}$ and $0 \leq i \leq 5$, $\deg_R(\nu + \chi_{\{v_i\}}) \geq 7$. Now, Theorem 4.2 implies that $\nu + \chi_{\{v_i\}}$ is equivalent to an effective divisor. This shows that ν_i , $1 \leq i \leq 3$ is extreme v_0 -reduced.

To finish the proof of the claim, it is enough to show that if ν is extreme v_0 -reduced divisor then $\nu \in \mathcal{E}$. Note that $\nu(v_1) = 0$ since otherwise $\nu - Qf \geq 0$ where $f = \chi_{\{v_0\}} + 3\chi_{\{v_1\}} + 2\chi_{\{v_2, v_3, v_4, v_5\}}$. Also, note that if $\nu(v_2) \geq 1$ and $\nu(v_3) \geq 1$, then $\nu - Qf \geq \chi_{\{v_1\}}$ where $f = \chi_{\{v_0, \dots, v_5\}}$. This shows that there exists $1 \leq i \leq 3$ such that $\nu = \nu_i$ or $\nu \sim \nu_i$.

The uniformity of (G, R) immediately follows from the fact that for all $\nu \in \mathcal{E}$, $\deg_R(\nu) = 4$.

For proving the fact that (G, R) is not reflection invariant, we apply a similar argument we used in the proof of Example 4.19. Let $\mathcal{P} = \{p_1, p_2, p_3\}$ be the same set as defined in Example 4.19. An easy computation shows that $p_1 = \frac{1}{3}(-2, -1, 4, -1, 4, -1)$, $p_2 = \frac{1}{3}(-2, -1, 7, -1, 1, -1)$ and $p_3 = \frac{1}{5}(-4, -3, 1, 7, 1, -3)$. For seeking a contradiction, assume there exists $v \in \mathbb{R}^6$ such that $-\text{Crit}(\Lambda) = \text{Crit}(\Lambda) + v$. Either there exist $\ell, \ell', \ell'' \in \Lambda$ such that $-p_1 = p_1 + \ell + v$, $-p_2 = p_2 + \ell' + v$ and $-p_3 = p_3 + \ell'' + v$, in this case $2(p_i - p_j) \in \Lambda$ for all $1 \leq i \neq j \leq 3$. Or, there exist $\ell, \ell' \in \Lambda$ and $\{i, j, k\} = \{1, 2, 3\}$ such that $-p_i = p_j + \ell + v$, and $-p_k = p_k + \ell' + v$, in this case $-p_j = p_i + \ell + v$ and we must have $-2p_k + p_i + p_j \in \Lambda$. Note that $\Lambda \subseteq \mathbb{Z}^6$, so an easy computation shows that none of the above cases occur. This proves that (G, R) is not reflection invariant. \square

Example 4.21. Suppose $R = (r_0, r_1, r_2) = (1, 2, 3)$. Let (G, R) be an arithmetical graph where G is a graph with vertex set $\{v_0, v_1, v_2\}$ such that the multiplicity of v_i is r_i and v_i is connected to v_j with $r_i r_j$ edges for all $0 \leq i \neq j \leq 2$. Then (G, R) is not uniform but it is reflection invariant.

Proof. We claim that $\nu_1 = -\chi_{\{v_0\}} + 3\chi_{\{v_1\}} + 2\chi_{\{v_2\}}$ and $\nu_2 = -\chi_{\{v_0\}} + \chi_{\{v_1\}} + 3\chi_{\{v_2\}}$ are the only extreme v_0 -reduced divisors. Suppose ν is an extreme v_0 -reduced divisor. Lemma 3.10 (ii) implies that $\nu(v_0) = -1$. It is not hard to see that $\nu(v_1) \leq 3$ and $\nu(v_2) \leq 3$, otherwise $\nu - Qf$ is effective where $f = \chi_{\{v_1, v_2\}}$ and $f' = \chi_{\{v_1\}} + 2\chi_{\{v_2\}}$ respectively. Moreover, if $D = -\chi_{\{v_0\}} + 2\chi_{\{v_1\}} + 3\chi_{\{v_2\}}$, then $D - Qf$ is effective where $f' = 2\chi_{\{v_1\}} + 3\chi_{\{v_2\}}$. Therefore the only possible extreme divisors are ν_1 and ν_2 . By running the generalized Dhar's algorithm on ν_1 and ν_2 , and applying Lemma 3.11, one can check that ν_1 and ν_2 are v_0 -reduced and therefore they are not equivalent to effective divisors. Note that the above computation shows that we already checked some of the different possible firing strategies in a run of the generalized Dhar's Algorithm on ν_1 and ν_2 .

So, we claim that if an arithmetical graph (G, R) has only two v_0 -reduced divisors then (G, R) is reflection invariant. Let Λ be the lattice spanned by Laplacian of (G, R) and \mathcal{E} be the set of extreme divisors of Λ .

By applying Lemma 3.6 and (ii) of Lemma 3.10, we conclude that $Ext(\Sigma(\Lambda)) = \{\nu + \ell : \ell \in \Lambda, \nu \in \mathcal{E}\}$. Corollary 2.14 implies $Crit(\Lambda) = \mathcal{P} + \Lambda$ where $\mathcal{P} = \{\pi(\nu + \vec{1}) : \nu \in \mathcal{E}\}$. Let ν_1 and ν_2 be the only extreme v_0 -divisors of (G, R) and $p_1 = \pi(\nu_1 + \vec{1})$ and $p_2 = \pi(\nu_2 + \vec{1})$. For proving the claim its enough to show that $-Crit(\Lambda) = Crit(\Lambda) + v$ where $v = -p_1 - p_2$. Assume $p \in Crit(\Lambda)$, therefore there exists $1 \leq i \leq 2$ and $\ell \in \Lambda$ such that $p = p_i + \ell$. Now, it is easy to see that $p_i + \ell + v = -p_j + \ell = -(p_j - \ell)$ where $j = -i + 3$ and $p_j - \ell \in Crit(\Lambda)$. This completes the proof of the claim.

So by a similar argument mentioned in proof of Example 4.20, (G, R) is reflection invariant. Since $\deg_R(\nu) = 11$ and $\deg_R(\nu') = 10$, we have $g_{\max} = 12$ and $g_{\min} = 11$. This shows that (G, R) is not uniform. \square

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